AFIT/GOA/ENS/97M-14

A COMPARISON OF CIRCULAR ERROR PROBABLE ESTIMATORS FOR SMALL SAMPLES

THESIS

Charles E. Williams, Captain, USAF

AFIT/GOA/ENS/97M-14

THESIS APPROVAL

Student: Charles E. Williams, Captain, USAF | Class: GOA-97M - 14

Title: A Comparison of Circular Error Probable Estimators for Small Samples

Defense Date: 4 Mar 97

Committee:	Name/Title/Department	Signature
Advisor	Edward F. Mykytka Associate Professor Department of Operational Sciences	Edward F. Mykytka
Reader	Paul F. Auclair Adjunct Assistant Professor Department of Operational Sciences	Paul F. Auclair

The views expressed in this thesis/dissertation are those of the author and do not reflect the official policy or position of the Department of Defense or the U.S. Government.

AFIT/GOA/ENS/97M-14

A COMPARISON OF CIRCULAR ERROR PROBABLE ESTIMATORS FOR SMALL SAMPLES

THESIS

Presented to the Faculty of the School of Engineering

of the Air Force Institute of Technology

Air Education and Training Command

In Partial Fulfillment of the

Requirements for the Degree of

Master of Science in Operations Analysis

Charles E. Williams, B.S., M.S.

Captain, USAF

March 1997

Approved for public release, distribution unlimited

Acknowledgments

I would like to extend my thanks to the people who made this paper possible. I begin with Dr. Ed Mykytka, whom I had the good sense to seek out as a thesis advisor; his time, patience and guidance were dearly appreciated. Capt. Bill Tongue's correspondence and 1993 thesis provided direction for the project, while the helpful suggestions contributed by Dr. Paul Auclair and Maj. Jim Miller made this paper much more insightful and readable. Most of all, the love, support and patience of my wife Chong kept me focused and gave me encouragement when I needed it most. Without this great abundance of help and motivation, this paper would not have been possible.

Charles E. Williams

Table of Contents

Page

List of Figures

Page

Page

List of Tables

AFIT/GOA/ENS/97M-n

Abstract

Several previous studies investigated the performance of competing circular error probable (CEP) estimators for small samples. This estimation is important in ICBM analysis because, due to expense, there are a limited number of ICBM test launches. In the most recent previous study (1993), Tongue considered five CEP estimators in a simulation test, attempting to determine the behavior of these estimators for populations of various bias, ellipticity, correlation, and sample size. In this paper, we build on Tongue's findings in three ways:

1.) The number of estimators compared is expanded to eight.

2.) Different factors and factor levels are used.

3.) In addition to analyzing simulated samples based on the population parameters used to create them, we sort the entire set of samples generated into subsets (sample analysis sets) based strictly on their sample statistics. Analysis conducted on these sample analysis sets models the real life situation of estimating CEP, given a small sample from an unknown population, using only sample statistics.

I. INTRODUCTION

Circular error probable, or CEP, is a statistical parameter used to describe the accuracy of ballistic projectiles. This widely used performance measure has a major impact on operational planning and analysis, as demonstrated by these three diverse examples:

1. The CEP of weapon systems such as ICBMs and the HELLFIRE antitank missile affects operational planning [Lewis (1994)].
2. CEP is used to describe the accuracy of surface to air missiles in the Joint Munitions Effectiveness Manual (JMEM) [Asner(1993)].
3. CEP influences weapon system performance in probabilistic combat simulation models.

The primary purpose of this thesis is to determine through a comparison study the effectiveness of a number of CEP estimators for small samples under various conditions. This determination is vital to intercontinental ballistic missile (ICBM) analysis since only a small number of test launches are conducted annually due to their great expense [Ethridge (1983), 1]. Furthermore, updates to ICBM components such as the missile guidance system (MGS) render flight test data which was collected prior to the update non-applicable. Therefore, analysis of ICBM test launches typically involves very small sets of data.

In this introductory chapter, we first present a general overview of basic CEP analysis concepts and terminology. We next discuss previous CEP estimator comparison studies. The chapter concludes with an outline of our experimental objectives and an overview of our experimental procedure.

Appendix A provides a key for the notation used in this thesis. It also lists the formulas used for calculating common sample statistics such as bias, ellipticity, and correlation.

1.1 An Overview of CEP Analysis

CEP is generally defined in relation to either the target or the mean point of impact. CEP defined with respect to the target is the radius of a circle, centered on the target, such that the probability of an impact landing inside the circle is 50% [Nelson (1988), 3]. CEP defined with respect to the mean point of impact is the radius of a circle, centered on the mean point of impact, such that the probability of an impact landing inside the circle is 50% [Thompson (1964), 1]. To distinguish between these two, we use "CEP" to refer to CEP with respect to the target while "CEP_{MPI}" refers to CEP with respect to the mean point of impact. In this study, the focus is on CEP, not CEP_{MPI}.

The objective in CEP analysis is to determine a reliable CEP estimate given a set of projectile impact points around a target. These ballistic impacts can be described using the following Cartesian coordinate system [Tongue (1993),1-1 and 1-2]:

1. The origin (0,0) corresponds to the target.
2. The crossrange miss distance (error) is measured on the X-axis (crossrange error axis).
3. The downrange miss distance (error) is measured on the Y-axis (downrange error axis).
4. The radial miss distance r_i for a point (x_i, y_i) is defined by

$$r_i = \sqrt{x_i^2 + y_i^2} \tag{1-1}$$

This Cartesian coordinate system is described in Figure 1.1 below:

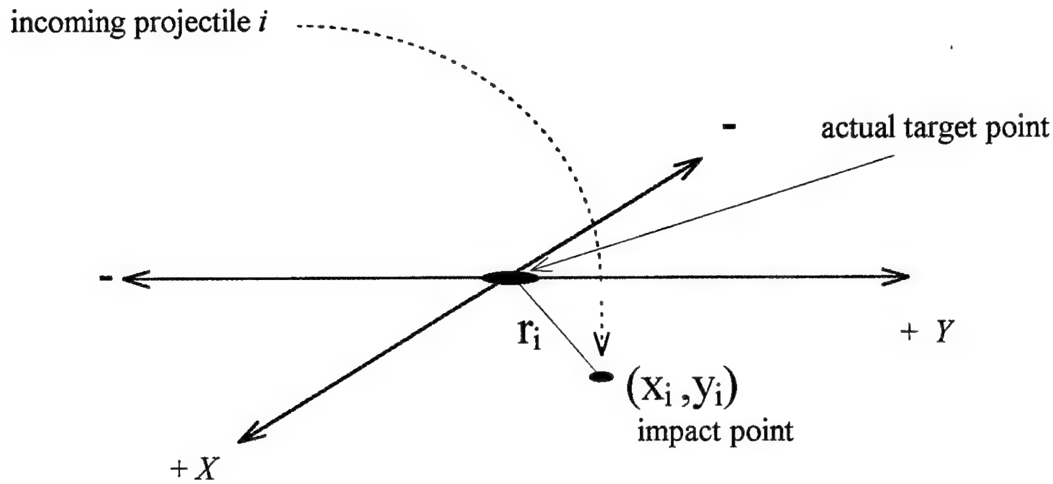


Figure 1.1 The Cartesian Coordinate System Used in CEP Analysis

The mean point of impact $(\overline{x}, \overline{y})$ is simply the coordinate of the sample crossrange mean and downrange mean.

For most CEP estimators, it is assumed that the downrange and crossrange miss distances follow a bivariate normal distribution. If the population parameters is known, the joint probability density function for this distribution is:

$$f(x,y) = \frac{1}{2\pi\, \sigma_x \sigma_y \sqrt{1-\rho^2}}\, e^{\left[\left(\frac{1}{2\left(1-\rho^2\right)}\right)\left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x \sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\}\right]} \tag{1-2}$$

Since this distribution forms a three dimensional shape, it is often described using "contour lines." Figure 1.2 portrays the contour lines corresponding to a given bivariate

normal distribution. Note that each contour line circumscribes a given proportion of the impact score population:

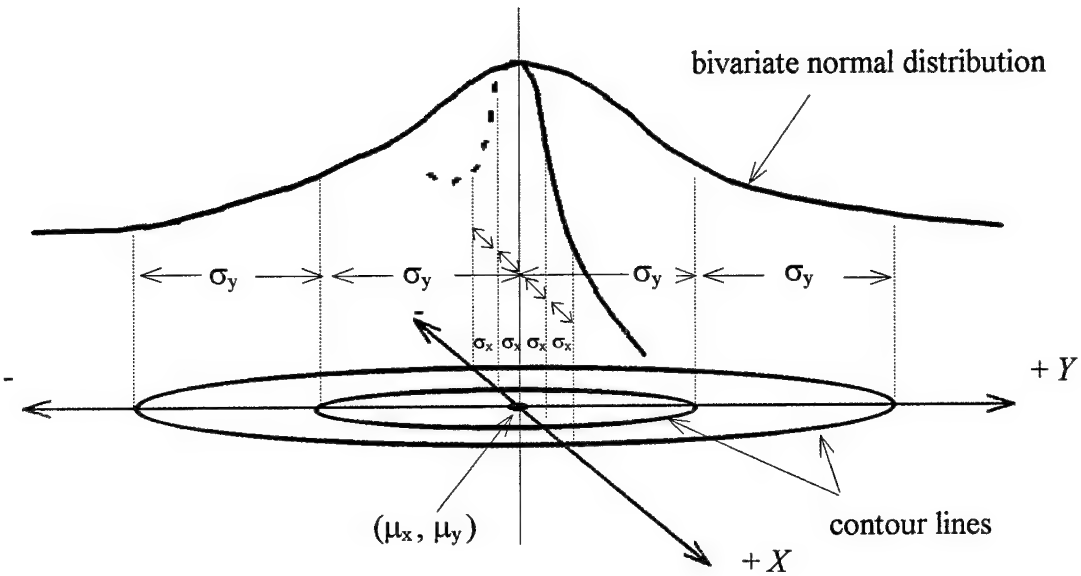


Figure 1.2 A Bivariate Normal Shape and its Resulting Contour Lines

Based on the assumption that the crossrange and downrange impact points follow a joint bivariate normal density function, if the population parameters are known, CEP is simply a circle of radius r such that the double integral below has a value of 0.5:

$$p(r) = \int_{-r}^{r} \int_{-\sqrt{r^2 - y^2}}^{\sqrt{r^2 - y^2}} \frac{1}{2\pi\,\sigma_x \sigma_y \sqrt{1-\rho^2}}\, e^{\left[\left(\frac{1}{2\left(1-\rho^2\right)}\right)\left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\,\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\}\right]}\, dxdy \tag{1-3}$$

Unfortunately, this integral has a closed form solution only under the special conditions described in Section 2.2. For the cases where the integral has no closed form

solution, the analyst has three choices:

1. Approximate the integral numerically to find this value for r.

2. Develop an empirical formula for estimating CEP such that when this CEP estimate is substituted in for r, p(r) will be approximately 0.5.

3. Apply a CEP estimator which is not based on the assumption that the impact points follow a bivariate normal distribution.

Chapter 2 provides an in-depth discussion of the CEP estimation methods considered in this thesis.

1.2 Previous CEP Comparison Studies

In this section we overview previous comparison studies which examined the performance of CEP estimators for "small" samples (sample size of 30 or less) under various conditions.

Elder (1986) presented one of the first comprehensive comparison studies of CEP estimators. Elder's experiment considered only bias and ellipticity. Although correlation can significantly influence the shape of the bivariate normal surface, Elder did not include it as a factor. In addition, Elder did not address the effect of sample size on the performance of the CEP estimators he compared.

Puhek's 1992 CEP estimator comparison study experiment expanded the factors to include bias, ellipticity, correlation, and sample size, but was limited in four ways:

i.) Only four CEP estimators were compared.

ii.) Only positive correlation values were used.

iii.) Estimate variance was not analyzed.

iv.) No replications were performed at any of the design points.

Tongue's 1993 comparison study is the most recent and the most complete. Like Puhek, the four factors of bias, ellipticity, correlation, and sample size were used in the comparison study. Like Puhek, only positive correlation values were considered and no replications were performed at any of the design points. Tongue's analysis in the comparison of five CEP estimators involved both the relative error and the variance of the estimates.

In this thesis, we build on the foundation of these earlier results by Elder (1986), Puhek (1992), and Tongue (1993). Six of the eight CEP estimators compared, as well as three of the measures of effectiveness used to compare the CEP estimators, come directly from Tongue's study.

1.3 Problem Statement

In this paper, the results of Elder (1986), Puhek (1992), and Tongue (1993) are extended in the following two ways:

1. Like the three previous studies, we conduct a simulation experiment in which CEP estimators are compared using simulated small samples from populations whose parameters are known. Our experimental design, however, includes some different factors and some different factor levels than those used by Elder, Puhek, or Tongue.

2. The three previous CEP estimator comparison studies developed recommendations for choosing a CEP estimation technique that required the analyst to know the values of the impact population parameters. But consider the example of ICBM missile accuracy analyst - the impact population parameters are unknown! In this study, we attempt to

determine the CEP estimator(s) of choice based strictly on sample, rather than population, parameters. By basing conclusions entirely on sample parameters, the real-life situation faced by an ICBM accuracy analyst is mirrored.

1.4 Experimental Overview

The term "design point' is used in this thesis to refer to a specific combination of levels for the experimental factors. In the simulation experiment, we first select relevant factors and then select levels for each of these factors to determine our design points.

The thesis experiment can be described as a four step process:

1. Ten simulated samples are generated at each design point. These replications at every design point make this thesis unique; previous comparison studies had no replications at any of the design points.

2. The eight CEP estimators being compared in this thesis are applied to the simulated output from each design point. These resulting CEP estimates are compared to the true CEP for the population from which the sample was generated. Bear in mind that in each case the parameters of the underlying bivariate normal distribution are determined by the design point and are thus known.

3. All of the sample data generated is next categorized into subsets (sample analysis sets) based on the sample statistics of sample size, sample bias, sample correlation, and the sample downrange to crossrange standard deviation ratio (s_x / s_y). These statistics are defined in Appendix A.

4. The eight CEP estimators considered in the thesis experiment are applied to each sample analysis set and the results are analyzed. When we analyzed the sample analysis

sets, the underlying bivariate normal distribution(s) for the data in each set was unknown! This comparison of CEP estimators based solely on sample statistics is unique to this study.

1.5 Objectives

The three fundamental objectives of our simulation experiment were to:

1. Provide a method to determine which CEP estimator(s) to use for a given particular set of conditions for a small sample (15 or less).
2. Compare the simulation experiment results based on the design points, where the population parameters are known, with the simulation experiment results based on the sample analysis sets, where the population parameters are unknown.
3. Verify the results from earlier CEP estimator comparison studies.

The remaining chapters all describe some aspect of the thesis experiment used to meet these three objectives. A reference for the mathematical formulas used for calculating each CEP estimator considered in the experiment is provided in Chapter 2. In Chapter 3 we develop the experimental approach, while in Chapter 4 we describe the experimental results. Based on these results, we present recommendations in Chapter 5.

II. CEP ESTIMATORS

This chapter briefly reviews the development of CEP estimation methods, presents a system for categorizing these methods, and describes the eight CEP estimators considered in this thesis in detail.

2.1 Background

One of the earliest CEP estimation aids was the publication of *Offset Circle Probabilities* by the Rand Corporation [Rand R-234 (1952)]. *R-234* consists of a set of lookup tables which allow for the estimation of CEP based upon CEP_{MPI} and the sample bias (defined in Appendix A). This set of tables provided an alternative to the tedious calculations involved in estimating CEP "by hand" in the pre-computer era. Pesapane and Irvine later developed the modified Rand-234 CEP estimator by fitting a wide range of values from these tables to a cubic polynomial [Pesapane and Irvine (1977)]. This estimator is not valid for certain cases of high ellipticity or high bias. The modified Rand R-234 CEP estimation method is described in detail in Section 2.3 of this thesis.

Harter (1960) and Kamat (1962) published early papers which contained algebraic CEP estimation techniques. The CEP estimators from Kamat's study assumed that the crossrange and downrange values followed a perfectly circular joint bivariate normal distribution centered at (0,0). This type of distribution implies that:

1. Crossrange and downrange means both equal zero.
2. Crossrange and downrange standard deviations are equal.
3. No correlation exists between the crossrange and downrange miss distances.

Harter allowed for ellipticity in his lookup tables, but maintained the assumption that the

bivariate normal distribution was centered around the target point. Unfortunately, in the study of actual projectiles, the distribution of impacts is often neither centered at the target point nor perfectly circular.

Grubbs' publication of *Approximate Circular and Noncircular Offset Probabilities of Hitting* [1964] spawned a number of CEP estimators. Two of the most common are the Grubbs-Patniak chi-square CEP estimator, which is described in Section 2.3, and the Grubbs-Patniak / Wilson-Hilferty CEP estimator. Elder [1986] cited that Strategic Air Command ICBM accuracy analysts relied on either one of these two Grubbs estimators or the modified Rand R-234 estimator for CEP calculations. Both of the Grubbs estimators are similarly calculated; they simply use different methods to calculate the inverse chi-square component in their algebraic formulas.

Ethridge [1983] introduced the robust, unbiased CEP estimator described in Section 2.3. Ethridge's estimator is unique in the respect that it is not based on the assumption that the impacts follow a joint bivariate normal distribution.

As computers and computer software developed in the 1980's, it became possible to numerically integrate the assumed bivariate normal distribution of ballistic impacts in a relatively short amount of time. Two examples of numerical CEP estimators are the direct numerical integration method (referred to as the "exact" method in some papers) and the correlated bivariate normal (CBN) method.

Tongue (1993) empirically derived an algebraic formula for predicting the relative error value when the CEP is estimated using the correlated bivariate normal (CBN) numerical integration CEP estimator. Tongue's preliminary analysis indicated that his

modified CBN estimator, described in Section 2.3, generally gave better estimates than the numerical CBN estimator [Tongue (1993), page 5-19]. If so, additional modifications of existing CEP estimators could be similarly developed.

2.2 CEP Estimator Categories

Smith (1982) classified CEP estimation methods into the five sets we use to categorize the CEP estimators considered in this thesis. We next describe each of these groups:

1. *Nonparametric Methods*: These CEP estimators make no assumptions regarding the underlying population of ballistic impacts. Smith uses the sample median as an example of a nonparametric CEP estimator.

Unlike nonparametric CEP estimators, the remaining four categories are based on the underlying assumption that the impacts follow some sort of theoretical joint probability distribution, usually the joint bivariate normal distribution.

2. *Closed-Form Integration of the Joint Bivariate Normal Density Function*: Closed form integration of the joint bivariate normal density function is only possible when the distribution is assumed be perfectly circular and centered at the origin. As previously stated, these conditions are often not met in the analysis of actual ballistic impacts. If these conditions are met, and σ represents the equal crossrange and downrange standard deviation values of the unbiased circular bivariate normal distribution, CEP is estimated by:

$$CEP = 1.1774\sigma. \tag{2-1}$$

3. *Algebraic Approximation CEP Estimation Methods*: The majority of CEP estimators

fall into this category. These approximation methods use an algebraic formula of impact population parameters (or their corresponding sample statistics) to estimate CEP.

4. *Numerical Integration of the Joint Bivariate Normal Density Function*: Unlike closed-form integration, numerical integration approximation techniques allow for correlation, bias, and ellipticity [Elder (1986), 1.2]. If these population values are known, numerical integration provides the most accurate approximation of CEP. When only sample statistics are known, these approximation methods can be used as estimators.

The author verified in a comparison of three different numerical integration methods (the correlated bivariate normal method, the Taylor series expansion method, and the direct numerical approximation method) that different numerical integration methods produce essentially identical results.

5. *Monte Carlo Sampling Methods*: These methods are used to calculate the probability of an impact landing inside a circle of a known radius. These methods, as described by Tongue (1993), essentially estimate CEP using simulation. Impact population parameters are estimated using sample statistics, and a set of "impacts" is simulated based on these parameters. An initial guess for the CEP value is determined, using for example an algebraic estimator. If half of the simulated impacts lie in a circle centered at the target with a radius equal to the CEP guess, the radius of this circle becomes the CEP estimate. If fewer than half land inside this circle, the CEP guess is increased by some increment and the process is repeated; if more than half of the simulated impacts land inside the circle, the CEP guess is decreased by some increment and the process repeated.

Smith (1982) concluded that for small samples Monte Carlo sampling techniques

are good tools for evaluating other CEP estimators, but impractical tools for actually estimating CEP because they use excessive computer calculation time [Smith (1982), 3]. Two additional limitations exist when trying to apply these techniques to small samples:

1. The sample estimates for the population parameters are not reliable for small samples.
2. For very small sample sizes, wide bands could exist between the points near the approximated CEP radius for the simulated samples.

2.3 Description of the CEP Estimators Compared in This Study

In the experimental phase of this research, eight individual CEP estimators were evaluated. We now describe how to calculate each of these estimators and demonstrate that they can provide different CEP estimates for the same data set. The practicing CEP analyst can use this section as a reference for CEP estimation methods.

1. The Sample Median CEP Estimator: Smith (1982) claimed that nonparametric CEP estimation methods are not suited for small samples. This assertion was tested by including the sample median of the radial miss distances in the thesis experiment.

The sample median CEP estimation value for a sample of size n, which we denote as *Smed*, is found by first sequentially ordering the radial miss distances from smallest to largest. Let $r_{[1]}$ to $r_{[n]}$ represent these ordered values. *Smed* is then found as follows:

$$Smed = \{\, r_{\left[\frac{n+1}{2}\right]} \text{ if n is odd, } \frac{1}{2}\left[r_{\left[\frac{n}{2}\right]} + r_{\left[\frac{n}{2}+1\right]} \right] \text{ if n is even} \} \qquad (2\text{-}2)$$

2. The Ethridge CEP Estimator: As mentioned in Section 1, Ethridge (1983) developed an algebraic estimator which did not use the underlying assumption that the impact population follows some bivariate normal distribution. Ethridge (1983) instead assumed

that the square root of the radial miss distances followed the logarithmic generalized exponential power distribution.

For a given set of impact coordinates, Ethridge incorporated the sample kurtosis and sample median statistics into his CEP estimator. To develop Ethridge's formula, we first define t_i and $\bar{t}$.

$$t_i = \ln(r_i) = \ln\left(\sqrt{x_i^2 + y_i^2}\right) \text{ for each sample point } (x_i, y_i) \tag{2-3}$$

Then average these t_i values to obtain $\bar{t}$:

$$\bar{t} = \frac{1}{n}\sum_{i=1}^{n} t_i \tag{2-4}$$

Sample kurtosis k is then computed by:

$$k = \frac{\sum_{i=1}^{n}(t_i - \bar{t})^4}{\left[\sum_{i=1}^{n}(t_i - \bar{t})^2\right]^2} \tag{2-5}$$

Next, let s_t^2 equal the sample variance of the t_i values, computed by:

$$s_t^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(t_i - \bar{t}\right)^2 \tag{2-6}$$

Hogg's unbiased estimator for the mean, which uses "weighted" values, is used to estimate the mean of the t_i values. For each coordinate in the sample, d_i is defined for computing "weight" w_i.

$$d_i = \text{maximum}\left\{1 - \left[0.03(k-3)^3 (t_i - Smed)^2 (s_t)^{-2}\right], 0.01\right\} \tag{2-7}$$

$$w_i = \frac{\frac{1}{d_i}}{\sum_{j=1}^{n}\frac{1}{d_j}} \tag{2-8}$$

These w_i values are then used to compute Hogg's estimator *u*:

$$u = \sum_{i=1}^{n} w_i t_i \tag{2-9}$$

Finally, Ethridge's CEP estimator, referred to hereafter simply as *Ethridge* in this study , is defined by:

$$\text{Ethridge} = e^{u} \tag{2-10}$$

It should be noted that the simulated sample values that are generated in the thesis experiment are all drawn from some type of bivariate normal distribution, even though *Ethridge* and the nonparametric *Smed* do not require this assumption.

3. The Modified Rand R-234 CEP Estimator: The algebraic Modified Rand R-234 CEP estimation formula is the result of a cubic polynomial regression analysis of the CEP table values found in the *Rand R-234* paper described in Section 2.1 [Pesapane and Irvine (1977)].

Before presenting the modified Rand R-234 CEP estimator, we first define it's components. For any bivariate normal distribution, the crossrange and downrange axes can be rotated to form two new axes such that the correlation of the distribution is zero in the newly formed coordinate system. The terms σ_S and σ_L represent the "short" and "long" standard deviation formulas when the axes are rotated to eliminate correlation:

$$\sigma_S = \sqrt{\frac{S_x^2 + S_y^2 - \sqrt{\left(S_x^2 + S_y^2\right) + 4\bar{\rho} S_x^2 S_y^2}}{2}} \tag{2-11}$$

$$\sigma_L = \sqrt{\frac{S_x^2 + S_y^2 + \sqrt{\left(S_x^2 + S_y^2\right) + 4\bar{\rho} S_x^2 S_y^2}}{2}} . \tag{2-12}$$

c = ellipticity in the rotated coordinate system = σ_S / σ_L . (2-13)

Let MR_{MPI} represent the modified Rand R-234 estimator for CEP_{MPI}, where:

$$MR_{MPI} = .563\ \sigma_L + .614\ \sigma_S \text{ (if } c > .25) \qquad (2\text{-}14)$$

Next, let v denote the bias expressed as a multiple of MR_{MPI}:

$$v = \sqrt{\bar{x}^2 + \bar{y}^2}\ /\ MR_{MPI} \quad . \qquad (2\text{-}15)$$

The modified Rand R-234 estimator is valid for c > .25 and v ≤ 2.2, the region over which Pesapane and Irvine regressed the Rand-234 tables [Pesapane and Irvine (1977), 3-5].

Letting *MRand* refer to the modified Rand R-234 CEP estimator,

$$MRand = [\ MR_{MPI}\ (\ 1.0039 - 0.0528v + 0.4786v^2 - 0.0793v^3\ )\ ]. \qquad (2\text{-}16)$$

4. The Valstar CEP Estimator: This algebraic estimator is similar to the modified Rand R-234 estimator, but has no exclusionary boundary conditions like *MRand*.

The values of σ_S, σ_L, and c in the equation that follows are computed identically to the *MRand* method, previously described in equations 2-1, 2-2, and 2-4. The equation for the estimator of the CEP with respect to the mean point of impact, denoted $Valstar_{MPI}$, is [Ethridge (1983), 12]:

$$Valstar_{MPI} = \begin{cases} 0.562\,\sigma_L + 0.615\,\sigma_S & \text{if } 0.369 < c \leq 1, \\ 0.675\,\sigma_L + \dfrac{\sigma_S}{1.2\sigma_L} & \text{if } 0 \leq c < 0.369 \end{cases} . \qquad (2\text{-}17)$$

The Valstar CEP estimator, henceforth referred to simply as *Valstar*, is:

$$Valstar = (Valstar_{MPI} + \bar{x}^2 + \bar{y}^2)^{1/2} . \qquad (2\text{-}18)$$

5. The Grubbs-Patniak Chi-Square CEP Estimator: Grubbs presented a CEP estimator based on the assumption that the x and y components of the radial miss distances follow a

normal distribution. If this assumption holds, then the sum of the squares of these r_i components follows a chi-square distribution [Grubbs (1963), 53]. The computation of the degrees of freedom of this chi-square distribution, denoted as *d*, requires the computation of variables defined as *m* and *v*. *Grubbs* is used to denote this CEP estimator in this study:

$$m = \bar{x}^2 + \bar{y}^2 + s_x^2 + s_y^2 \tag{2-19}$$

$$v = 2(s_x^4 + 2\,\bar{\rho}^2 s_x^2 s_y^2 + s_y^4) + 4(\bar{x}^2 s_x^2 + 2\,\bar{x}\,\bar{y}\,\bar{\rho}\, s_x s_y + \bar{y}^2 s_y^2) \tag{2-20}$$

$$d = 2m^2 v^{-1} \tag{2-21}$$

$$\text{Grubbs} = \sqrt{\frac{v\left[\text{chisq}^{-1}(.5, d)\right]}{2\,m}}\,. \tag{2-22}$$

6. The Rayleigh Distribution CEP Estimator: This simple algebraic estimator is based on the assumption that the radial miss distances follow a Rayleigh distribution [Ethridge (1983), 10-11]. In this study, we refer to this estimator simply as *Rayleigh*. The formula for *Rayleigh* is:

$$\text{Rayleigh} = .9394\,\bar{r} \tag{2-23}$$

where $\bar{r}$ = the mean of the radial miss distances.

7. The Direct Numerical Integration CEP Estimator: This estimator approximates CEP using the following double integral formula for p(r):

$$p(r) = \int_{-r}^{r} \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{1}{2\pi S_x S_y \sqrt{1-\bar{\rho}^2}}\, e^{\left[\left(\frac{1}{2\left(1-\bar{\rho}^2\right)}\right)\left\{\left(\frac{x-\bar{x}}{S_x}\right) - 2\bar{\rho}\left(\frac{(x-\bar{x})(y-\bar{y})}{S_x S_y}\right) + \left(\frac{y-\bar{y}}{S_y}\right)\right\}\right]} dxdy$$

(2-24)

This integral describes the volume under the surface formed by a bivariate normal distribution, centered at $(\bar{x}, \bar{y})$, with correlation $\bar{\rho}$ and standard deviations s_x and s_y with respect to the crossrange and downrange axes. To find the numerical estimate for the CEP, we want p(r) to equal 0.5. The corresponding value for *r*, approximated using a numerical root finding method, is the direct numerical integration CEP estimator. It is denoted by the word *Numerical* in this thesis.

8. Tongue's Modified CBN CEP Estimator: This estimator is presented in Tongue [1993, pages 6-1 through 6-6]. Using regression analysis, Tongue developed a formula for predicting the relative error value when the CEP is estimated using the correlated bivariate normal (CBN) numerical integration CEP estimator (or any numerical integration CEP estimator, since different numerical integration methods produce nearly identical results). Tongue's CBN relative error estimator is a function of the sample size, scaled sample bias, and sample ellipticity (where sample ellipticity is defined by equation A-10 of this thesis).

The numerical CBN CEP estimator is based on the equivalent polar coordinate form of the double integral formula for p(r) given in equation 2-24 [Elder (1986), 3-6]:

$$p(r) = \frac{1}{2\pi S_x S_y \sqrt{1-\bar{\rho}^2}} \int_0^{2\pi} \int_0^{r} r\, e^{\left(-A r^2 + B r + C\right)}\, dr d\theta\,, \tag{2-25}$$

where:

$$A = \frac{1}{2\left(1-\bar{\rho}^2\right)} \left[\frac{\sin^2\theta}{S_x{}^2} - \frac{2\bar{\rho}\sin\theta\cos\theta}{S_x S_y} + \frac{\cos^2\theta}{S_y{}^2} \right], \tag{2-26}$$

$$B = \frac{1}{2\left(1-\bar{\rho}^2\right)} \left[\frac{\bar{x}\sin\theta}{S^2} - \frac{\bar{\rho}\,\bar{x}\cos\theta + \bar{\rho}\,\bar{y}\sin\theta}{S_x S_y} + \frac{\bar{y}\cos\theta}{S_y{}^2} \right], \tag{2-27}$$

$$\text{and } C = \frac{1}{2\left(1-\bar{\rho}^2\right)}\left[\frac{\bar{x}^2}{S_x{}^2} - \frac{2\,\bar{\rho}\,\bar{x}\,\bar{y}}{S_x S_y} + \frac{\bar{y}^2}{S_y{}^2}\right]. \tag{2-28}$$

To find the numerical estimate for the CEP, we want p(r) = 0.5. Like other numerical integration CEP estimation methods, the value for *r*, which we denote as *CBN*, can be approximated using a numerical root finding method.

The notation $\hat{bias}$ and $\hat{ellip}$ are used respectively in this thesis for the sample bias and sample ellipticity. Let $\hat{bias}_s$ denote the scaled sample bias:

$$\hat{bias}_s = \hat{bias}\left[\sqrt{\frac{S_x{}^2 + S_y{}^2}{2}}\right]^{-1} \tag{2-29}$$

Next, we compute Tongue's relative error estimator $R\hat{e}$:

$$R\hat{e} = 0.171833 - 0.009784n - 0.037707\,\hat{bias}_s - 0.\ 150628\hat{ellip}$$

$$+\ 0.002045(n)(\hat{bias}_s) + 0.007488(n)(\hat{ellip}) + 0.019014(\hat{bias}_s)(\hat{ellip})$$

$$+\ 0.116385(\hat{ellip})(\bar{\rho}) - 0.006714(\hat{ellip})(\bar{\rho})(n) \tag{2-30}$$

Tongue's Modified CEP estimator, denoted as *TMCBN*, is then computed by:

$$TMCBN = CBN\,(\,1 - R\hat{e}\,)\,. \tag{2-31}$$

We close this chapter by presenting an example which demonstrates that these eight estimators can (and usually do) produce differing CEP estimates from the same set of sample data. Consider the following sample of six impact coordinates:

$$\text{Sample} = \begin{bmatrix} 165 & 140 \\ 37 & -60 \\ 217 & -207 \\ 81 & -223 \\ 113 & 119 \\ 257 & -155 \end{bmatrix}$$

The resulting sample statistics are:

$\bar{x}$ =	145.0	$\bar{\rho}$ =	-0.42
$\bar{y}$ =	-150.67	$\bar{r}$ =	214.71
s_x =	83.56	$\hat{bias}$ =	209.11 ($2.881\bar{\sigma}$)
s_y =	59.60	n =	6.0
$\bar{\sigma}$ =	72.58		

The eight CEP estimates for this data are:

Smed =	226.82	*Grubbs* =	214.11
Ethridge =	194.52	*Rayleigh* =	201.70
MRand =	218.30	*Numerical* =	216.02
Valstar =	224.35	*TMCBN* =	220.20

Summarizing our first two chapters, we have discussed basic CEP analysis concepts and described each estimator used in our thesis experiment. In our remaining chapters, we shift the focus from these preliminary topics to our simulation experiment design and the analysis of its results.

III. DESCRIPTION OF THE SIMULATION EXPERIMENT

We designed our experiment to compare the eight CEP estimators presented in Chapter 2. We begin this chapter by examining relationships that exist between our experimental factors. We next discuss the specific factor levels used to form our design points and show how we constructed the sample analysis sets, which are based entirely on sample statistics. The chapter concludes with a discussion of the measures of effectiveness that we used to assess the relative performance of the eight CEP estimators considered.

3.1 Relationships Between the Experimental Factors

We assumed in our experiment that impacts follow a bivariate normal distribution. In selecting the levels for our factors, we strived to meet two goals. First, to provide a comprehensive study, we wanted the design points to adequately represented a wide range of bivariate normal distributions. Second, we wanted to avoid using redundant design points. Several relationships that exist between the factors used in our experiment enabled us to meet these two goals in our design.

Consider any bivariate normal distribution which describes the points of impact of incoming projectiles. Suppose this distribution is centered at (μ_x, μ_y) in the first quadrant of the Cartesian coordinate system representing the downrange and crossrange miss distances, with the angle of rotation above the positive X-axis represented by θ. Next, consider a bivariate normal distribution in the third quadrant with identical shape parameters, centered at $(-\mu_x, -\mu_y)$, and an identical angle of rotation θ formed below the negative X-axis. Because of the symmetry depicted in Figure 3.1, the radius of a circle

containing half of the points from each distribution must be the same. Hence, the two distributions will have identical CEP values. This same relationship holds for the second and fourth quadrants.

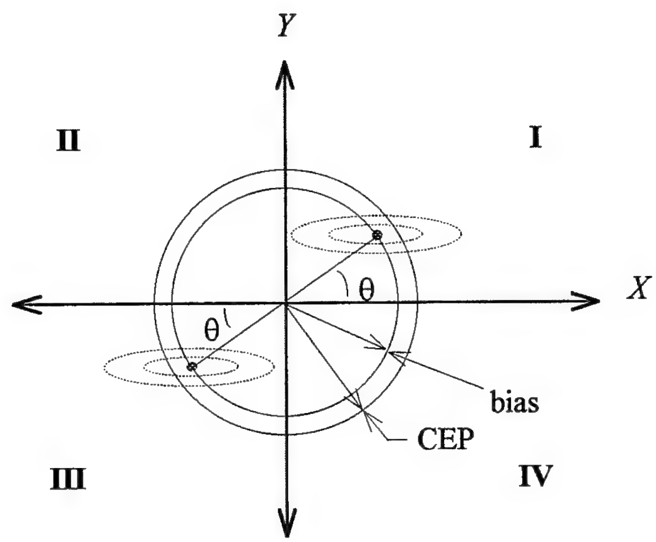


Figure 3.1 Identically Shaped Bivariate Normal Distributions With Equal CEP Values

Consider again any bivariate normal distribution which describes the points of impact of incoming projectiles, centered at (μ_x, μ_y) in the first quadrant with the angle of rotation θ. Let k be the correlation of this bivariate normal distribution. Now consider a bivariate normal distribution in the second quadrant, centered at $(-\mu_x, \mu_y)$, rotated θ above the negative X-axis, with identical shape parameters except for correlation, which is $-k$. Because of the shape and bias of the 2 distributions, the radius of a circle containing half of the points from each distribution must be the same; therefore, both have an identical CEP value. An example of this relationship is displayed in Figure 3.2:

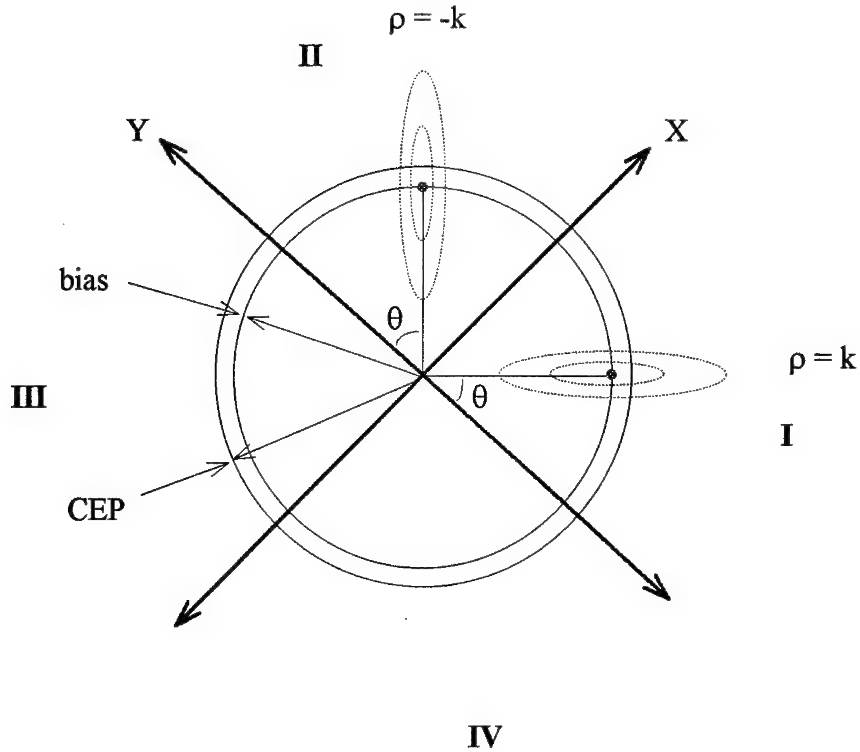


Figure 3.2 An Example of Bivariate Normal Distributions in Quadrants I and II With Equal CEP Values

Note that we display the axes rotated at an angle of 45^0 in Figure 3.2 and other figures in this paper. This rotation is due to a software limitation, and is not intended to make any point. Microsoft Word 6.0 was used to process this document, and this software only allows the construction of ellipses aligned completely parallel or vertical to the horizontal.

From these illustrations, we conclude that for any bivariate normal shape in the second, third, or fourth quadrant, there exists a bivariate normal distribution in the first quadrant with an identical CEP value. Thus, using mean points of impact entirely from the first quadrant, we can adequately represent all possible bivariate normal distributions,

provided we use both positive and negative correlation levels.

We now present special symmetries which exist along the axes. These symmetries allowed us to eliminate certain redundant design points in our experimental design. An example of each case is presented in Figure 3.3.

Case I: Consider the special case of a bivariate normal distribution with correlation k which is centered upon either the X or the Y axis. Due to their symmetrical shapes, a second bivariate normal distribution, centered at the same point along the axis, identical in every way except for the a correlation value of $-k$, must have the same CEP as the original distribution. Thus, for the special case of bivariate shapes centered along one of the axes, we can adequately represent all possible bivariate normal shapes using only positive correlation values.

Case II: Any bivariate normal shape along the negative X-axis centered at $(-\mu_x, 0)$ with correlation k must have the same CEP value as an identically shaped bivariate normal distribution centered at $(\mu_x, 0)$ with correlation k or $-k$. This relationship also holds true for the Y-axis. We can therefore adequately represent all possible bivariate shapes along the X-axis or Y-axis using only bivariate normal distributions centered along the nonnegative portion of the axis.

Case III: Consider any bivariate normal shape centered along the positive Y-axis at (0, p), with crossrange and downrange standard deviation values of a and b respectively. Now consider a second bivariate normal shape centered along the X-axis at (p , 0) with an identical correlation, an identical bias value, and crossrange and downrange standard deviation values of b and a respectively. Because of their location on the X and Y axes,

identical bias, and a shape which is identical except that second distribution is rotated ninety degrees, the radius of a circle containing half of the points from each distribution must be the same. Hence, these two distributions must have an identical CEP. It follows from this relationship that as long as the σ_y / σ_x levels are reciprocals (such as 2 and 1/2) we can use design points centered along the positive X-axis to represent all of the bivariate normal shapes centered along either the X or Y axis.

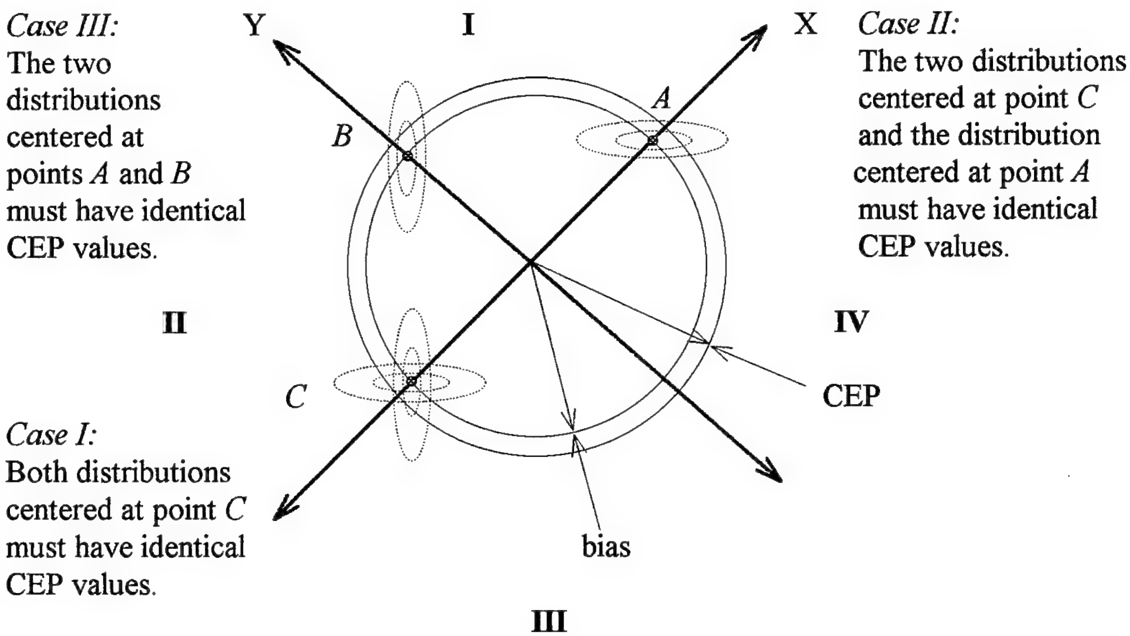
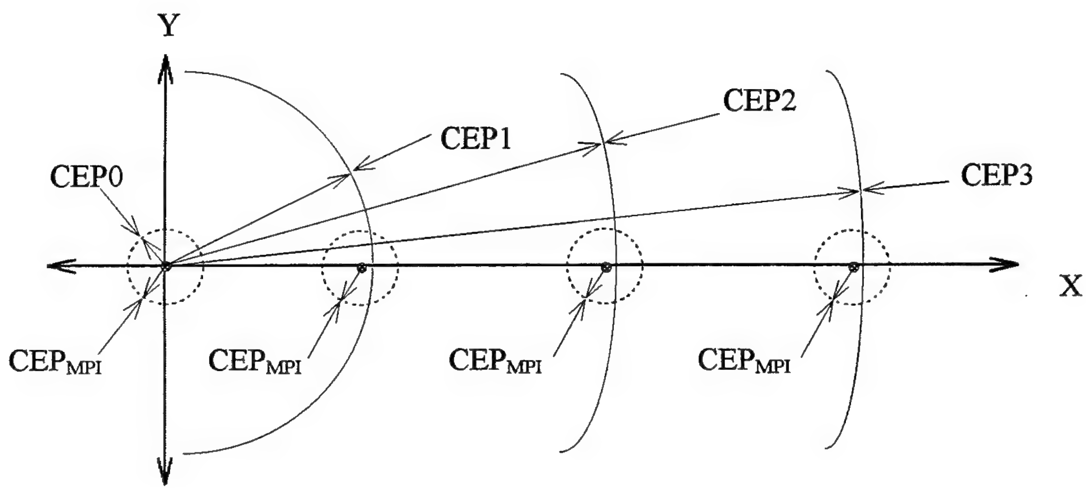


Figure 3.3 Bivariate Normal Shapes With Equal CEP Values

We conclude this section on experiment factor relationships by exploring the relationship between bias and the CEP/CEP$_{MPI}$ ratio. If this ratio is very large, it indicates that there is a "tight" cluster of impacts that are relatively far away from the intended target. All other things being equal, one logical way to increase the probability of hitting

close to the target if this case occurs is to intentionally aim off of the target in such a way that the tight cluster is centered on target. Thus, the statistical parameters of the tight cluster, such as CEP_{MPI}, are probably more significant than the parameter of CEP for this circumstance.

Consider the examples in Figure 3.4. In each case the shape of the bivariate normal surface, and thus the CEP_{MPI}, is identical. The only difference in the distributions is the center point and bias values. The example illustrates the fact that for any bivariate normal shape, as the bias increases, so does the CEP/CEP_{MPI} ratio. It follows that large CEP/CEP_{MPI} ratios occur only when the bias is very near the CEP value.



$CEP0/CEP_{MPI} = 1$; $CEP1/CEP_{MPI} = 6$; $CEP2/CEP_{MPI} = 13$; $CEP3/CEP_{MPI} = 20$

Figure 3.4 CEP/CEP_{MPI} Increases as Bias Increases

The CEP/CEP_{MPI} ratios for the experimental design points, which are listed in Appendix C, range from 1.000 to 4.277. The largest bias level used in the experiment is 0.999 CEP; to produce larger CEP/CEP_{MPI} ratios, larger bias levels would be necessary.

3.2 The Simulation Experiment Factors

In this section, we present each of our five experimental factors and the levels used in our experimental design for each factor.

1. *Correlation*: In earlier studies, Tongue(1993) and Puhek(1992) used only bivariate normal shapes centered along the positive X-axis. In our design, all of our bivariate normal distributions are centered within the first quadrant, but not all along the X-axis. For the design points not along the X-axis, both positive and negative correlation values are necessary to adequately represent equivalent bivariate normal distributions in all of the other three quadrants for the reasons described in Section 3.1. Using both positive and negative correlation levels is an important distinction between our experimental setup and the earlier studies; Tongue and Puhek used only positive correlation values, while Elder(1986) did not use correlation as a factor at all.

Correlation values can range from -1.0 to 1.0. For our experiment, the five uniformly spaced correlation levels of -0.8, -0.4, 0, 0.4, and 0.8 were selected. Due to the symmetry described in Section 3.1, only positive correlation values were used for design points centered on the X-axis.

2. *The Ratio of* σ_y / σ_x : Elder (1986), Puhek (1992), and Tongue (1993) used ellipticity as a factor. In this study, we instead use the ratio of downrange standard deviation to crossrange standard deviation (σ_y / σ_x). The reason for this choice is demonstrated in the following example:

Consider the following two impact populations below. Both have an identical CEP of 100 and are identical in every way (including ellipticity) except for their σ_x ,σ_y ,

and σ_y / σ_x values:

	p	μ_x	μ_y	bias	σ_x	σ_y	σ_y/σ_x	ellipticity
#1	-0.4	85	0	85	108.9	21.8	0.2	0.2
#2	-0.4	85	0	85	13.2	65.9	5.0	0.2

Population #1: Population #2 :

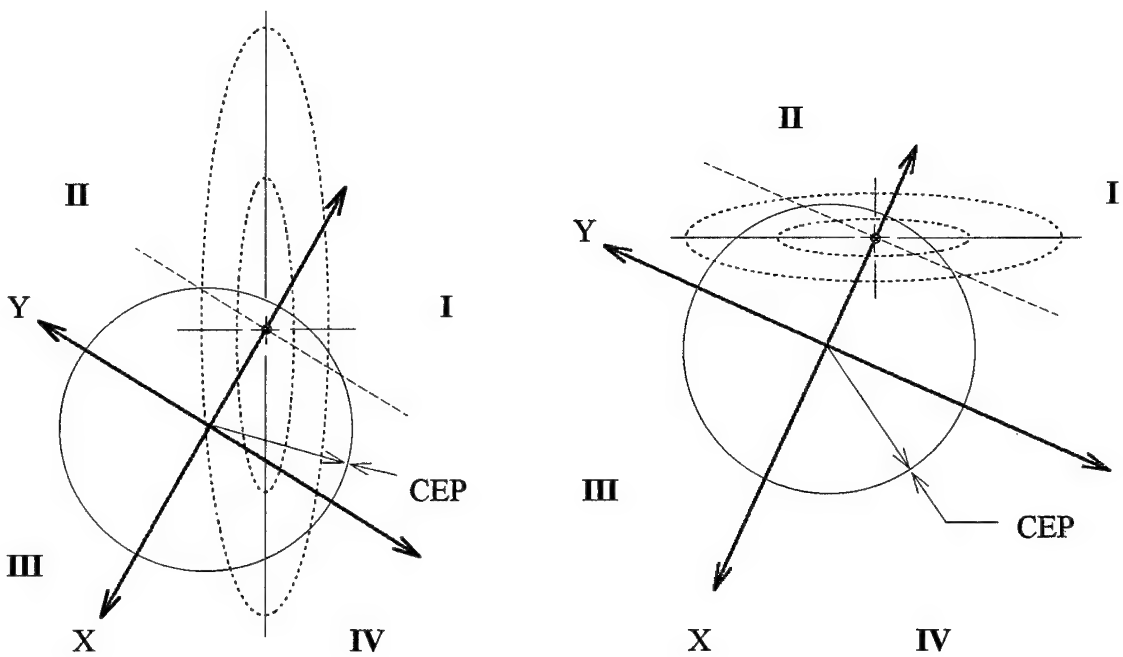


Figure 3.5 Contour Lines of Example Impact Populations #1 and #2

Observe, as suggested in Figure 3.5, that the contours of the bivariate normal distribution in population #1 will enclose much larger areas than the corresponding contours of the bivariate normal distribution in population #2 due to the larger crossrange and downrange variances in population #1. This example shows that two bivariate normal

distributions which have an identical mean point of impact and ellipticity can require significantly different values for their larger and smaller standard deviation values in order to achieve an equal CEP. By using σ_y / σ_x as a factor instead of ellipticity, and by using reciprocal levels for σ_y / σ_x (such as the values of five and 0.2 used in the example), we force the inclusion of both such bivariate normal shapes.

The five levels for σ_y / σ_x used in this study are 0.2, 0.6,1 , 1.667, and 5. The levels of 0.2, 0.6, and 1 are the low, middle, and high values from the five ellipticity levels used in Tongue's 1993 study. The levels of 5 and 1.667 are included as the reciprocals of 0.2 and 0.6 respectively.

3. *Bias*: In order to maintain generality, we use the following scaling factor to normalize our bias level values:

$$\sigma = \sqrt{\frac{\sigma_x^{\,2} + \sigma_y^{\,2}}{2}}. \tag{3-1}$$

Tongue(1993) and Puhek(1992) considered bias levels range between zero and 2σ inclusively. In this study, we extend the tested bias range to include 4σ by using the bias levels 0, 0.5σ, 1.0σ, 2.0σ, and 4.0σ.

4. *Sample Size*: Small sample sizes are of particular interest in this study. Tongue concluded in his study that for sample sizes greater than ten, the CBN CEP estimator (or any other numerical integration CEP estimator) returned the least biased CEP estimate and recommended that future studies focus on sample sizes of ten or less [Tongue(1993), 7-7]. Based on this recommendation, we chose three sample size levels that were less than ten: three, six, and nine. We also included a sample size which was greater than ten, fifteen, to

test Tongue's assertion with our design point results. Note that Tongue's assertion was based on results where the generating population was known. Using a sample size level of fifteen also enabled us to analyze sample analysis sets, based strictly on sample statistics, with a sample size which was greater than ten.

5. *Rotation Angle θ*: The uniformly spaced levels of 0^0 , 30^0, and 60^0 were selected for our final experimental factor, the angle of rotation θ above the positive X-axis. As previously stated, (μ_x, μ_y) for all of our design points lies in the first quadrant. Thus, our design points are centered according to their bias levels either along the X-axis or along imaginary lines rotated 30^0 and 60^0 above the x-axis.

A complete list of the 275 design points used for each sample size is provided in Appendix C. The experimental factors and the levels used for each factor are summarized in the table below:

Table 3.1 Combinations of These Values are Used to Form the Design Points

Sample Size	3,	6,	9,	15	
Bias	0,	0.5σ,	σ,	2σ,	4σ
Correlation (only nonnegative values used along the X axis)	-0.8,	-0.4,	0,	0.4,	0.8
σ_y/σ_x	0.2,	0.6,	1,	1.667,	5
Rotation angle (degrees) above the X-axis:	0^0,	30^0,	60^0		

Once again, it is emphasized that this was the first CEP estimator comparison study which used replications at each of the design points. Only one simulation run was accomplished at each design point in the earlier studies; in our experiment, ten simulation runs were performed at each design point. These multiple runs allowed us to compute

sample statistics at each design point. In addition, we could apply the measures of effectiveness discussed in Section 3.7 at each design point.

3.3 The Design Point Templates:

For ease in comparing the eight CEP estimators, especially with regards to the sample analysis set results, we created a design wherein each population had an identical CEP of 100. Thus, in our experiment, the expected CEP value for each sample, regardless of design point, was 100.

In this section, we explain how we created MathCAD (1995) templates (programs) to find parameter values such that each design point would have a CEP of 100.

As stated in Chapter 1, if the population parameters of the underlying bivariate normal distribution are known, then CEP is best approximated using numerical integration. To accomplish this calculation using direct numerical integration requires the following population parameters:

1. The crossrange and downrange means.
2. The crossrange and downrange standard deviations.
3. The correlation between the corresponding crossrange and downrange values.

For each design point, however, the following information is known:

1. The correlation between the corresponding crossrange and downrange values.
2. σ_y/σ_x.
3. Bias.
4. The angle of rotation θ above the positive X-axis.

Correlation is a factor. Therefore, for any design point p_0, we have a correlation

value directly assigned. If we can define μ_y, σ_y, and σ_x in terms of μ_x for p_0, we will have only one unknown (μ_x) which we can numerically solve for.

First, we use rotation angle θ to define μ_y in terms of μ_x :

$$\tan\theta = \mu_y/\mu_x \quad \Rightarrow \quad \mu_y = \mu_x \tan\theta. \tag{3-2}$$

We can similarly state σ_x in terms of σ_y using σ_y/σ_x. This ratio is set at one of five levels for each design point. Let σ_y/σ_x equal k_1 for p_0. We can algebraically restate the equation as:

$$\sigma_y = k_1 \sigma_x \tag{3-3}$$

The bias of p_0, is assigned one of five levels, each of which can be represented as a constant k_2 multiplied by σ. By definition, we can also represent the bias of p_0 in terms of μ_x and μ_y. Below, we set these two equivalent expressions for the bias of p_0 equal to each other, then substitute in the values from equations 3-2 and 3-3:

$$\text{bias of } p_0 = \sqrt{\mu_x^2 + \mu_y^2} = k_2\sqrt{\frac{\sigma_x^2 + \sigma_y^2}{2}}$$

$$\Rightarrow \sqrt{\mu_x^2 + \mu_x^2 \tan^2\theta} = k_2\sqrt{\frac{\sigma_x^2 + k_1^2\sigma_x^2}{2}} \quad \Rightarrow \quad \mu_x\sqrt{1+\tan^2\theta} = k_2\sigma_x\sqrt{\frac{1+k_1^2}{2}}$$

$$\Rightarrow \quad \mu_x = k_2\,\sigma_x\,\frac{\sqrt{\frac{1+k_1^2}{2}}}{\sqrt{1+\tan^2\theta}} \qquad\qquad \Rightarrow \quad \mu_x = k_2\,\sigma_x\,\frac{\sqrt{\frac{1+k_1^2}{2}}}{\sec\theta}$$

$$\Rightarrow \quad \sigma_x = \mu_x \frac{\sec\theta}{k_2\sqrt{\frac{1+k_1^2}{2}}} \tag{3-4}$$

Substituting for σ_x in equation 3-3,

$$\sigma_y = k_1 \mu_x \frac{\sec\theta}{k_2 \sqrt{\frac{1+k_1^2}{2}}} \tag{3-5}$$

Using equations 3-3 and 3-5, we constructed MathCAD (1995) templates to find parameters such that each design point would have a CEP = 100 within a tolerance of 0.01. An example of one of these templates is provided in Appendix B. Note the accuracy of the CEP values listed in the template output for this example.

3.4 The Sample Analysis Sets

Recall that for each sample size the term "sample analysis set" refers to a set of generated sample data whose sample bias, sample correlation, and sample crossrange to sample downrange standard deviation ratio fall into given specific ranges.

For each of these three sample statistics, we used ranges for the sample analysis sets which were roughly centered around the design factor levels. We first display the sample bias ranges used:

Population Bias Levels	0 $\quad 0.5\sigma$	1.0σ	2.0σ	4.0σ
Sample Bias Ranges	$[0, 0.75\bar{\sigma}]$	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	$(>2.75\bar{\sigma})$

Although it would appear that the first sample bias range of $[0, 0.75\bar{\sigma}]$ would apply to more outcomes than the other three ranges (since the expected value of the sample bias for design points where the bias equals zero or 0.5σ falls into this range), high variance within the small sample sizes used resulted in a relatively uniform allocation in

each of the four sample bias categories.

The sample correlation ranges were centered directly around the population correlation levels as follows:

Population Correlation Levels	-0.8	-0.4	0	0.4	0.8
Sample Correlation Ranges	[-1.0,-0.6)	[-0.6,-0.2)	[-0.2,0.2]	(0.2,0.6]	(0.6,1.0]

Finally, the sample analysis set s_x / s_y ranges were centered around the population σ_x / σ_y levels, except for the open-ended range (> 2.5):

Population σ_x / σ_y Levels	0.2	0.6	1.0	1.667	5.0
Sample s_x / s_y Ranges	(0, 0.4)	[0.4, 0.8)	[0.8, 1.25]	(1.25, 2.5]	(> 2.5)

All of the 2,750 samples generated for each sample size were sorted into sample analysis sets for the comparison of the eight CEP estimators based strictly on sample statistics. The combinations used to form the sample analysis sets for each sample size are summarized below in Table 3.2; a complete list of the sample analysis sets for each sample size is presented in Appendix D.

Table 3.2 Combinations Used to Form the Sample Analysis Sets for Each Sample Size

Sample Bias Range	$[0,0.75\bar{\sigma}]$, $(0.75\bar{\sigma},1.25\bar{\sigma}]$, $(1.25\bar{\sigma}, 2.75\bar{\sigma}]$, $(>2.75\bar{\sigma})$
$\bar{\rho}$ Range	[-1.0, -0.6), [-0.6, -0.2), [-0.2, 0.2], (0.2, 0.6], (0.6, 1.0]
s_x/s_y Range	(> 2.5), (1.25, 2.5], [0.8, 1.25], [0.4, 0.8), (< 0.4)

3.5 The Sample Generator Program

In this section, we describe the program used to generate the simulated samples. In our program design, we took advantage of the fact that we can form two new axes (X'

and Y') and "rotate" them such that no correlation is present in the new coordinate system. As described by Tongue [(1993), pages 3-12 and 3-13], the angle of rotation θ is:

$$\theta = \frac{1}{2}\arctan\left(\frac{2\,\rho\,\sigma_x\,\sigma_y}{\sigma_x - \sigma_y}\right) \tag{3-6}$$

The values for μ_x', μ_y', σ_x', and σ_y' in this new rotated axis system are:

$$\mu_x' = \mu_x \cos\theta + \mu_y \sin\theta \tag{3-7}$$

$$\mu_y' = -\mu_x \sin\theta + \mu_y \cos\theta \tag{3-8}$$

If $\sigma_x > \sigma_y$:

$$\sigma_x' = \sqrt{\frac{\sigma_x^2 + \sigma_y^2 + \sqrt{\left(\sigma_x^2 + \sigma_y^2\right) + 4\,\rho\,\sigma_x\sigma_y}}{2}} \tag{3-9}$$

$$\sigma_y' = \sqrt{\frac{\sigma_x^2 + \sigma_y^2 - \sqrt{\left(\sigma_x^2 + \sigma_y^2\right) + 4\,\rho\,\sigma_x\sigma_y}}{2}} \tag{3-10}$$

If $\sigma_x \leq \sigma_y$:

$$\sigma_x' = \sqrt{\frac{\sigma_x^2 + \sigma_y^2 - \sqrt{\left(\sigma_x^2 + \sigma_y^2\right) + 4\,\rho\,\sigma_x\sigma_y}}{2}} \tag{3-11}$$

$$\sigma_y' = \sqrt{\frac{\sigma_x^2 + \sigma_y^2 + \sqrt{\left(\sigma_x^2 + \sigma_y^2\right) + 4\,\rho\,\sigma_x\sigma_y}}{2}} \tag{3-12}$$

Our sample generation program, listed in Appendix E, is written in the MODSIM programming language [CACI (1995)]. This program first transforms the input μ_x, μ_y, σ_x, and σ_y values into μ_x', μ_y', σ_x', and σ_y'. Next, for the input sample size, random x' and y' values are selected from the bivariate normal distribution.

To apply our CEP estimators to the simulated samples created, the (x', y') values generated were next transformed back into their corresponding (x, y) values in the original X and Y coordinate system using these two equations:

$$x = x'\cos\theta - y'\sin\theta \tag{3-13}$$

$$y = x'\sin\theta + y'\cos\theta \tag{3-14}$$

The program requires the sample size to be manually input. Each line of an input file for the program consists of a given design point's crossrange mean , downrange mean, crossrange standard deviation, and downrange standard deviation values, plus the number of runs desired for the given design point. The output for each run consists of the sample size, sample correlation, sample crossrange mean , sample downrange mean, sample crossrange standard deviation, sample downrange standard deviation, sample median, sample mean radial error, and the Ethridge CEP estimate.

To verify that the program produces reliable output, a file containing 20 varied design points, each with a population CEP of 100, was created. For this test run, the sample size was set to 1000. The output from this file showed that the generated points accurately reflected the input bivariate normal shape characteristics. The input and output files are presented in Tables 3.3 and 3.4. One can observe the accuracy of the program by comparing the μ_x, μ_y, σ_x, σ_y, and ρ values of the input file with their corresponding $\bar{x}$, $\bar{y}$, s_x, s_y, and $\bar{\rho}$ sample statistic values in output file. In Table 3.5, the relative error between each of these five parameters and their corresponding sample statistics is presented for comparison.

Table 3.3 The Input File Used to Verify the Sample Generator Program

Repetitions	μ_x	μ_y	σ_x	σ_y	ρ
1	73.6	42.5	98.5	19.7	-0.8
1	35	60.6	103.2	20.6	-0.8
1	60.6	35	85.7	51.4	-0.8
1	60.6	35	20.6	103.2	-0.8
1	42.5	73.6	19.7	98.5	-0.8
1	35	60.6	23.2	116	-0.8
1	73.6	42.5	98.4	19.7	-0.4
1	60.6	35	115.3	23.1	-0.4
1	35	60.6	63.1	63.1	-0.4
1	60.6	35	45.6	75.9	-0.4
1	60.6	35	20.2	101.2	-0.4
1	42.5	73.6	19.7	98.4	-0.4
1	35	60.6	23	115.1	-0.4
1	73.6	42.5	103.4	20.7	0.4
1	60.6	35	71.8	71.8	0.4
1	35	60.6	55.4	92.4	0.4
1	35	60.6	111	22.2	0.8
1	73.6	42.5	71.8	71.8	0.8
1	60.6	35	58.1	96.8	0.8
1	42.5	73.6	21.4	106.9	0.8

Table 3.4 The Resulting Output for the Input File in Table 3.3:

Sample Size	$\bar{x}$	$\bar{y}$	s_x	s_y	$\bar{\rho}$	*Smed*	$\bar{r}$	*Ethridge*
1000	75.325	42.295	98.565	19.643	-0.802	99.704	117.11	103.65
1000	34.498	60.515	101.1	20.193	-0.794	98.668	113.43	104.06
1000	59.041	35.225	83.738	49.721	-0.793	97.794	107.46	96.107
1000	60.449	36.033	21.19	104.74	-0.811	99.471	115.81	105.81
1000	41.964	78.586	19.745	101.52	-0.792	99.655	119.3	104.58
1000	34.7	63.228	22.822	115.07	-0.792	98.182	117.93	100.82
1000	75.101	42.788	98.271	19.575	-0.393	102.28	115.89	100.2
1000	59.621	35.147	115.24	22.842	-0.36	99.953	116.34	96.897
1000	35.595	61.578	62.365	63.807	-0.403	99.809	103.41	90.807
1000	60.747	33.893	45.372	74.934	-0.397	97.562	102.26	91.853
1000	60.526	34.018	19.94	101.39	-0.33	100.23	112.96	102.6
1000	42.493	70.016	19.919	96.985	-0.418	98.714	113.7	99.475
1000	35.524	59.577	22.593	111.4	-0.359	97.781	113.94	95.348
1000	74.591	42.183	102.95	20.386	0.4124	99.626	116.38	97.135
1000	60.58	35.625	73.966	71.307	0.4239	100.35	109.14	91.89
1000	34.881	63.196	54.846	90.963	0.3873	97.364	110.23	90.924
1000	33.602	60.234	110.48	22.43	0.8078	100.35	117.27	104.17
1000	72.226	40.452	71.417	71.331	0.7956	96.699	111.02	90.079
1000	59.333	33.028	59.428	97.898	0.8166	104	114.26	95.239
1000	42.688	76.026	22.07	110.3	0.8062	98.492	119.13	95.821

Table 3.5 Relative Error (RE) of the Corresponding Sample Statistics For the Input Population Parameters in Table 3.3

μ_x	$\bar{x}$	RE	μ_y	$\bar{y}$	RE	ρ	$\bar{\rho}$	RE
73.6	75.325	0.0234	42.5	42.295	0.0048	-0.8	-0.802	0.0027
35	34.498	0.0143	60.6	60.515	0.0014	-0.8	-0.794	0.0078
60.6	59.041	0.0257	35	35.225	0.0064	-0.8	-0.793	0.0088
60.6	60.449	0.0025	35	36.033	0.0295	-0.8	-0.811	0.0136
42.5	41.964	0.0126	73.6	78.586	0.0677	-0.8	-0.792	0.0095
35	34.7	0.0086	60.6	63.228	0.0434	-0.8	-0.792	0.0106
73.6	75.101	0.0204	42.5	42.788	0.0068	-0.4	-0.393	0.0188
60.6	59.621	0.0162	35	35.147	0.0042	-0.4	-0.36	0.1005
35	35.595	0.017	60.6	61.578	0.0161	-0.4	-0.403	0.0075
60.6	60.747	0.0024	35	33.893	0.0316	-0.4	-0.397	0.0068
60.6	60.526	0.0012	35	34.018	0.0281	-0.4	-0.33	0.1758
42.5	42.493	0.0002	73.6	70.016	0.0487	-0.4	-0.418	0.0443
35	35.524	0.015	60.6	59.577	0.0169	-0.4	-0.359	0.1023
73.6	74.591	0.0135	42.5	42.183	0.0075	0.4	0.4124	0.031
60.6	60.58	0.0003	35	35.625	0.0178	0.4	0.4239	0.0598
35	34.881	0.0034	60.6	63.196	0.0428	0.4	0.3873	0.0319
35	33.602	0.0399	60.6	60.234	0.006	0.8	0.8078	0.0097
73.6	72.226	0.0187	42.5	40.452	0.0482	0.8	0.7956	0.0055
60.6	59.333	0.0209	35	33.028	0.0563	0.8	0.8166	0.0208
42.5	42.688	0.0044	73.6	76.026	0.033	0.8	0.8062	0.0077

σ_x	s_x	RE	σ_y	s_y	RE	Median	*Smed*	RE
98.5	98.565	0.0007	19.7	19.643	0.0029	100	99.704	0.003
103.2	101.1	0.0203	20.6	20.193	0.0198	100	98.668	0.0133
85.7	83.738	0.0229	51.4	49.721	0.0327	100	97.794	0.0221
20.6	21.19	0.0286	103.2	104.74	0.0149	100	99.471	0.0053
19.7	19.745	0.0023	98.5	101.52	0.0307	100	99.655	0.0035
23.2	22.822	0.0163	116	115.07	0.008	100	98.182	0.0182
98.4	98.271	0.0013	19.7	19.575	0.0063	100	102.28	0.0228
115.3	115.24	0.0005	23.1	22.842	0.0112	100	99.953	0.0005
63.1	62.365	0.0116	63.1	63.807	0.0112	100	99.809	0.0019
45.6	45.372	0.005	75.9	74.934	0.0127	100	97.562	0.0244
20.2	19.94	0.0129	101.2	101.39	0.0018	100	100.23	0.0023
19.7	19.919	0.0111	98.4	96.985	0.0144	100	98.714	0.0129
23	22.593	0.0177	115.1	111.4	0.0321	100	97.781	0.0222
103.4	102.95	0.0043	20.7	20.386	0.0152	100	99.626	0.0037
71.8	73.966	0.0302	71.8	71.307	0.0069	100	100.35	0.0035
55.4	54.846	0.01	92.4	90.963	0.0156	100	97.364	0.0264
111	110.48	0.0047	22.2	22.43	0.0104	100	100.35	0.0035
71.8	71.417	0.0053	71.8	71.331	0.0065	100	96.699	0.033
58.1	59.428	0.0229	96.8	97.898	0.0113	100	104	0.04
21.4	22.07	0.0313	106.9	110.3	0.0318	100	98.492	0.0151

3.6 Generating the CEP Estimates From the Sample Generator Output Data

Given any set of MODSIM generator program output, our MathCAD CEP estimator template listed in Appendix F computes the CEP value for each of the eight CEP estimators considered in our experiment. In addition, the template also calculates the measures of effectiveness discussed in Section 3.7.

The generated output for each design point, where the underlying population parameters were known, and for each sample analysis set, where the underlying population parameters were unknown, were processed through this template.

The run time for the MathCAD CEP estimator template with a 100 MHz. Pentium processor equipped P.C. was normally less than two minutes per design point, or around five minutes for a thirty element sample analysis set.

In retrospect, one aspect that we failed to address in our MathCAD CEP estimator template was the identification of design points which were invalid for *MRand.* This estimator is only valid for certain cases described in Section 2.3. By processing every design point for *MRand*, it was difficult to fairly evaluate this estimator.

3.7 Measures of Effectiveness

We use the term "measures of effectiveness", or MOEs, to refer to the criteria used to distinguish between CEP estimators. Before we present the MOE which we selected for evaluating our experimental results, we first discuss the MOEs used in the earlier studies of Elder (1986), Puhek (1992), and Tongue (1993).

Elder and Puhek reached their conclusions by identifying the CEP estimator with the least relative error (RE) for the majority of their design point results, where RE is

defined:

$$RE = |\text{ actual CEP - estimated CEP }| / \text{actual CEP}. \quad (3\text{-}15)$$

Unfortunately, comparing the CEP estimators using only RE ignores the important characteristic of variance.

Consider the two distributions of CEP estimates in Figure 3.6 below. Based strictly on minimizing bias or relative bias, distribution #1 would be "best". However, the high variance of distribution #1 versus distribution #2 illustrates the need for a more effective MOE to compare these two CEP estimate distributions.

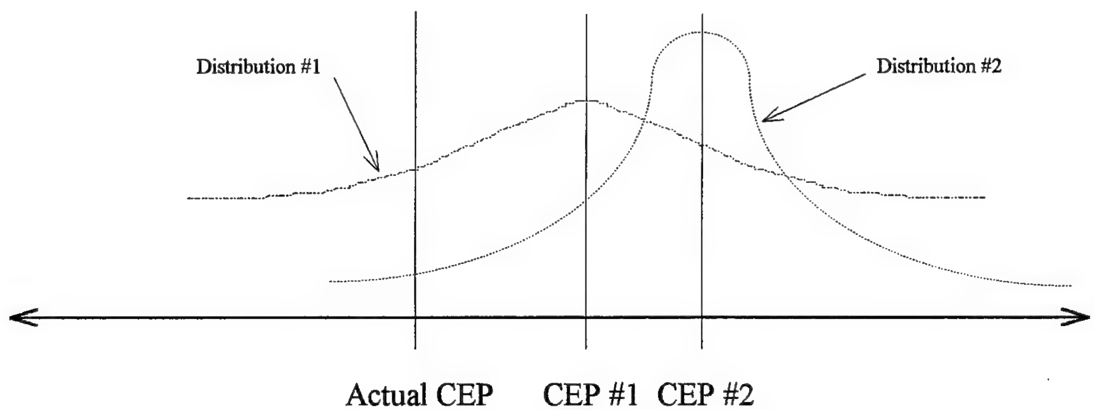


Figure 3.6 Distribution #1 has Less Bias but Higher Variance than Distribution #2

Instead of determining which CEP estimator had the least relative error for the majority of his design point results, Tongue considered the *mean* relative error (MRE) of each CEP estimator in his 1993 study. In addition, Tongue considered two other MOEs. The first of these was to compare which estimator had the least variance among it's MRE value; we use VRE hereafter to refer to this MOE. Tongue's second additional MOE was

the mean square of the relative error, hereafter denoted as MSRE, which is defined in either of the following manners:

$$MSRE = \frac{1}{n}\sum_{i=1}^{n} RE_i^{\,2} = MRE^2 + VRE \tag{3-16}$$

In our study, we computed the results for all three MOEs from Tongue's study for every design point and sample analysis set result. In our actual analysis, however, we relied on the MSRE instead of MRE or VRE for the following two reasons:

1. MSRE combines the effectiveness characteristics of both MRE and VRE.
2. As seen in Appendices I and J, the estimator with the best MSRE score for a given set of conditions was also virtually always best for either MRE or VRE, and usually both.

To demonstrate how an MSRE difference between estimators relates to the relative error between the estimators, we offer the plot in Figure 3.7:

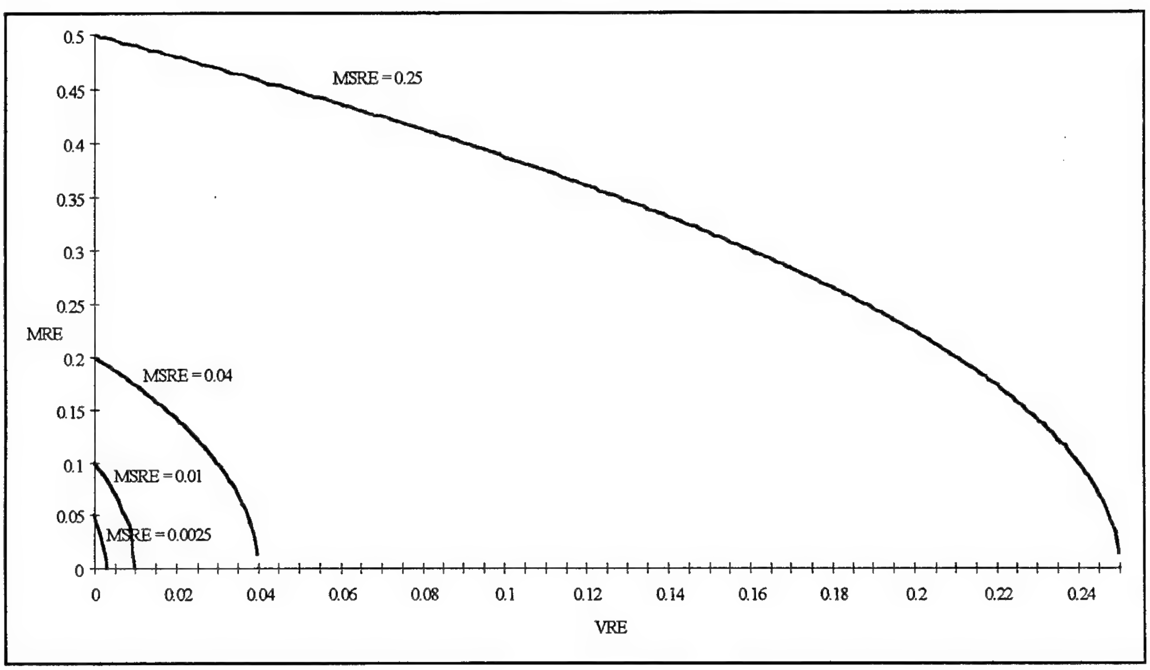


Figure 3.7 Relationship of MSRE to MRE and VRE

Consider a CEP estimator which always yields an estimate with an error of exactly five percent. In terms of RE, this difference would equal 0.05. In terms of MSRE, this difference would correspond to 0.05 squared, or 0.0025. As seen in Figure 3.7, an MSRE of amount X equates to a relative error of $\sqrt{X}$ if the variance is zero. Therefore, if the MSRE is known, the square root of the MSRE provides an approximation for the MRE. This approximation for the MRE multiplied by the actual parameter being estimated (in our case CEP) yields an approximation of the actual error one could expect based on the known MSRE. We hereafter use AE to refer to this MSRE-based approximation of error.

We use the MOEs of MSRE and AE extensively in the analysis of our experimental results in the remaining two chapters. In Chapter 4, we outline these experimental results, while in Chapter 5 we present recommendations based on these results.

IV. THE THESIS EXPERIMENT RESULTS

In this chapter, we present the results of the experiment described in Chapter 3. We followed an identical experimental process for both the design points and sample analysis sets to analyze the differences and similarities in their results. The experimental protocol consisted of:

1. Presenting overall observations of the resulting data.
2. Investigating which experimental factors appeared to be most significant.
3. Investigating combinations of those factors determined to be the most significant.

In addition to comparing our design point results against our sample analysis set results, we also compared our design point results with those of previous studies.

4.1 Convergence Deviations for the Numerical and TMCBN CEP Estimators

In our MathCAD CEP estimator template described in Section 3.6, we designated a precision tolerance of 0.01 for the calculations of all eight CEP estimators. The implication of this tolerance for the root finding algorithm employed by *Numerical* and *TMCBN* was that r was approximated such that $0.49 \leq p(r) \leq 0.51$. We discovered design points and sample analysis sets, however, where our root finding algorithm would not converge at this tolerance. These deviations occurred primarily at sample size three.

These deviant design points and sample analysis sets are listed in Table 4.1. When the root finding algorithm of *Numerical* and *TMCBN* did not converge using a tolerance of 0.01, we systematically tried larger tolerances of 0.1, 0.2, 0.5, and 1.0. Any design point or sample analysis set that would not converge at a tolerance of 1.0 was conceded as non-convergent:

Table 4.1 Design Points/Sample Analysis Sets Where a Tolerance of Other Than 0.01 Was Used for *Numerical* and *TMCBN*

Sample Size	Design Pt. Reference #	Tolerance Used
6	230	0.1
3	9	0.2
3	20	0.1
3	24-25	0.1
3	30	0.1
3	33-34	0.1
3	40	0.5
3	55	0.1
3	57	0.1
3	69	0.1
3	80	0.1
3	91	0.1
3	95-98	0.1
3	111	0.2
3	112	0.1
3	114	0.1
3	117	0.1
3	120	0.2
3	122-124	0.1
3	131	*
3	132	0.1
3	142-143	0.1
3	145	0.1
3	147	0.2
3	151	0.5
3	152	0.1
3	155	0.5
3	159-160	0.5
3	161	*
3	162	0.1
3	165-166	0.1
3	170	0.1
3	174	0.1
3	175	0.2
3	179	0.2
3	181-182	0.2

(continued)

Sample Size	Sample Analysis Set Reference #	Tolerance Used
6	80	0.1
3	1	*
3	3-4	*
3	5	0.2
3	22	*
3	24-25	0.1
3	26-28	*
3	30	0.1
3	47	*
3	50	0.1
3	52	*
3	53-54	0.1
3	55	0.2
3	71	0.2
3	72	*
3	73	0.1
3	74	*
3	75	0.2
3	76	*
3	77	0.2
3	79	*
3	80	*
3	86	1.0
3	95	0.2
3	96	1.0
3	97	0.1
3	98	0.1
3	99-100	*

NOTE

* - Indicates no convergence for the given design point or sample analysis set.

Sample Size	Design Pt. Reference #	Tolerance Used
3	183-187	0.1
3	188	0.2
3	189	0.1
3	190	0.2
3	191-201	0.1
3	203-205	0.1
3	210-214	0.1
3	215	0.2
3	216-220	0.1
3	221	0.2
3	222-223	0.1
3	224	0.5
3	225-226	*
3	227	0.2
3	228	*
3	229	0.1
3	230	*
3	231	0.1
3	232	0.5
3	233-234	0.1
3	235	1.0
3	236-239	0.1
3	240	*
3	241-244	0.1
3	245	*
3	246-248	0.1
3	249	*
3	250-251	0.1
3	252	0.5
3	253-355	0.1
3	256	*
3	257-263	0.1
3	264	0.2
3	265-269	0.1
3	270	1.0
3	271	*
3	272	0.5
3	273	0.1
3	274	*
3	275	0.1

TOTALS FOR SAMPLE SIZE = 6

Tolerance Used	Design Points	Sample Analysis Sets
0.1	1	1

TOTALS FOR SAMPLE SIZE = 3

Tolerance Used	Design Points	Sample Analysis Sets
0.1	99	8
0.2	14	6
0.5	9	0
1.0	2	2
*	12	16

NOTE

* - Indicates no convergence for the given design point or sample analysis set.

Note from Table 4.1 that all of the non-convergent cases occurred exclusively when the sample size equaled three. These non-convergent cases appear related to the σ_y/σ_x and s_y/s_x parameters, based on the patterns displayed in Table 4.2:

Table 4.2 Relationship Between σ_y/σ_x, s_y/s_x, and Non-Convergent Cases

σ_y/σ_x	Number of Non-Convergent Design Points
0.2	5
0.6	0
1	1
1.667	2
5	4

s_y/s_x	Number of Non-Convergent Sample Analysis Sets
(< 0.4)	1
[0.4,0.8)	5
[0.8,1.25]	2
(1.25,2.5]	6
(> 2.5)	2

Tongue (1993) had similar convergence problems with his numerical CBN CEP estimator in his study. He determined that convergence problems occurred when the sample statistics from a highly elliptical population reflected a predominately circular distribution or vice-versa [Tongue (1993), page 4-16]. As seen in Table 4.2, our non-convergent design points tend to originate from populations of extreme ellipticity; we believe that sample statistics reflecting a predominately circular distribution were generated from these design points. Our non-convergent sample analysis sets, on the other hand, tend to have s_y/s_x values that indicate a predominately circular distribution. We believe that these non-convergent sample analysis sets contain the output from one or more of the non-convergent design points.

Because a relatively large number of *Numerical* and *TMCBN* estimates were computed using a tolerance of less than 0.01 at sample size three, the MOE scores of these two estimators at this sample size were distorted. Given more time, we would have

investigated the performance of *Numerical* and *TMCBN* at sample size three for only those design points and sample analysis sets that converged at a tolerance of 0.01.

4.2 The Design Point Results

Now that we have outlined the convergence problems for the two numerical estimators considered, we next present out design point results.

The total experimental output was voluminous, as evidenced by the design point and sample analysis set MSRE results presented in Appendices G and H. Our first attempt at interpreting this large volume of data was to determine for all design point and sample analysis set cases which estimator had the most "best finishes" (lowest scores) for the MOEs of MRE, VRE, and MSRE. These values are recorded in Appendices I and J respectively. While these numbers give an indication of how the considered CEP estimators performed, they can also be misleading.

Suppose that estimators A and B are being compared for a given case containing thirty design points. Suppose further that when the MSRE values are compared, estimator A "wins" twenty times while estimator B "wins" only ten times. Estimator A appears to be the better estimator. What is unknown from these numbers, however, is the *magnitude* for each of the respective "wins." Suppose that when A wins the margin between A and B is very narrow, while when B wins the margin is very great. Clearly a more effective tool is needed to compare these estimators than just the number of "wins."

We decided for most cases that the best way to display the performance of the competing estimators was to construct cumulative distribution function (CDF) plots relating the MSRE results of the competing estimators to cumulative probability.

When comparing CDFs, a distribution A "stochastically dominates" another distribution B for a given set of continuous random numbers X if the following condition is met [Clemen (1996), 238]:

$$F_A(x) \leq F_B(x) \; \forall \, x \in X \tag{4-1}$$

Stochastic dominance is indicated graphically by the plot of the CDF of A always lying equal to or below the plot of the CDF of B:

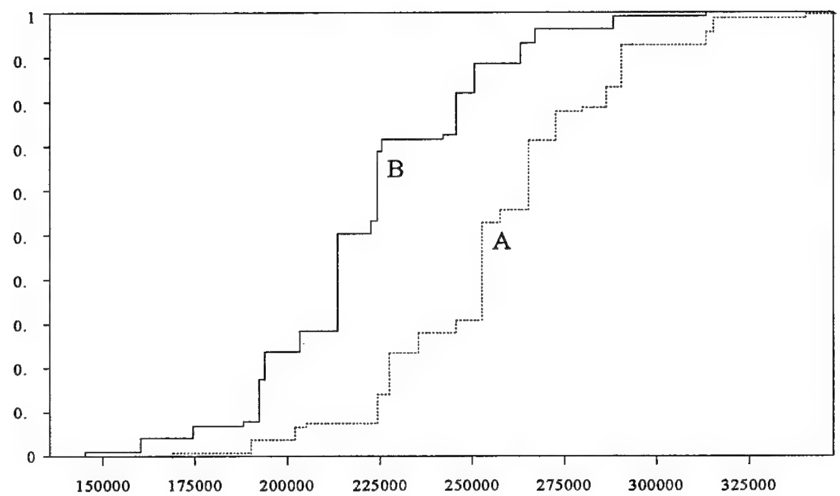
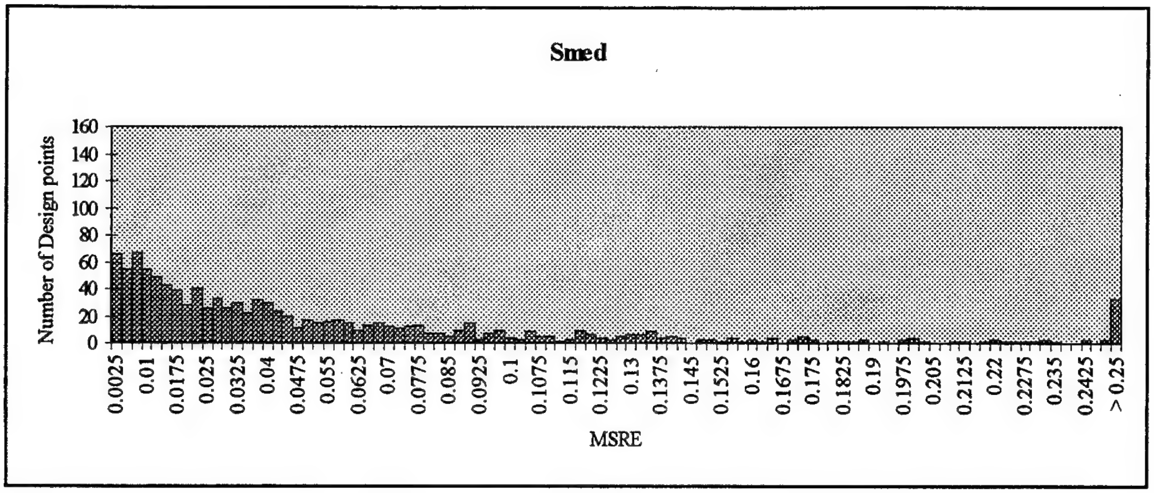
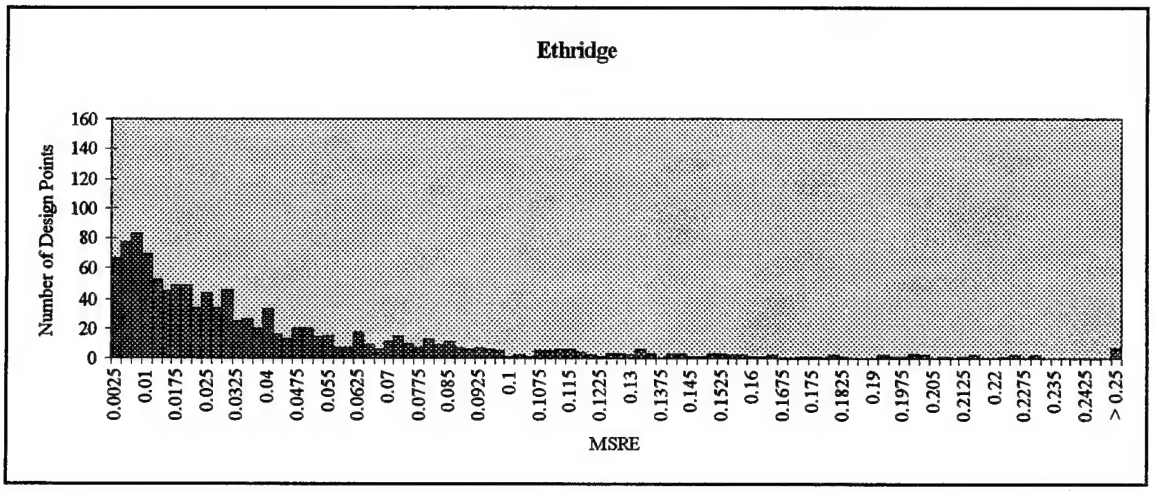


Figure 4.1 Distribution A Stochastically Dominates Distribution B

In our situation, good performance is indicated by *low* MSRE scores. Therefore, an estimator that stochastically dominates another can be considered inferior.

We begin our presentation of the design point results by displaying histograms for each CEP estimator considered. An estimator that has primarily low MSRE values shows large bars on the left side of the graph; conversely, large bars on the right hand side of the graph indicate that the given estimator had a large number of design points with a relatively high CEP estimation error. Note that the far right bar refers to an open-ended range of all design points where the MSRE was > 0.25:





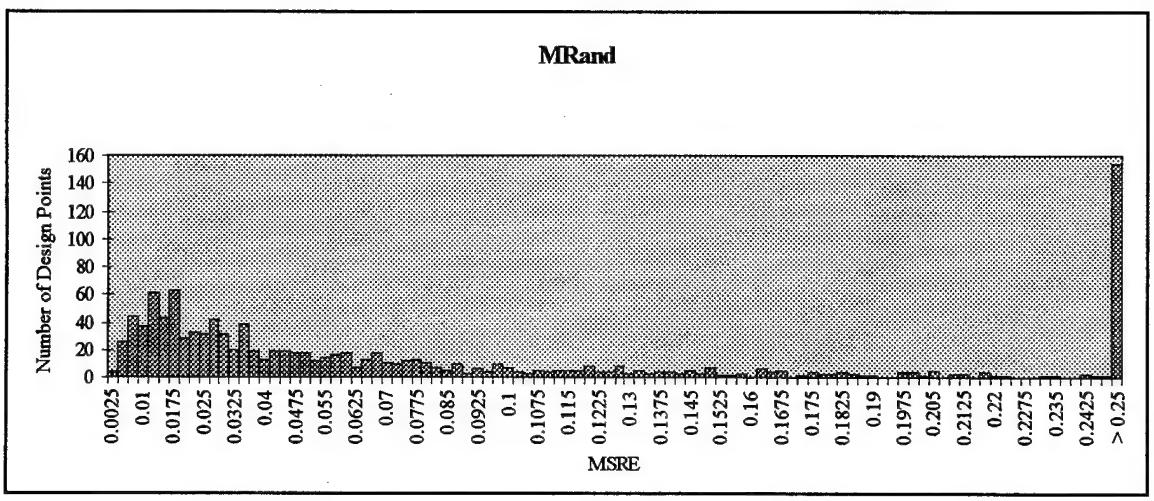


Figure 4.2 Overall Histograms Based on the Design Point Results

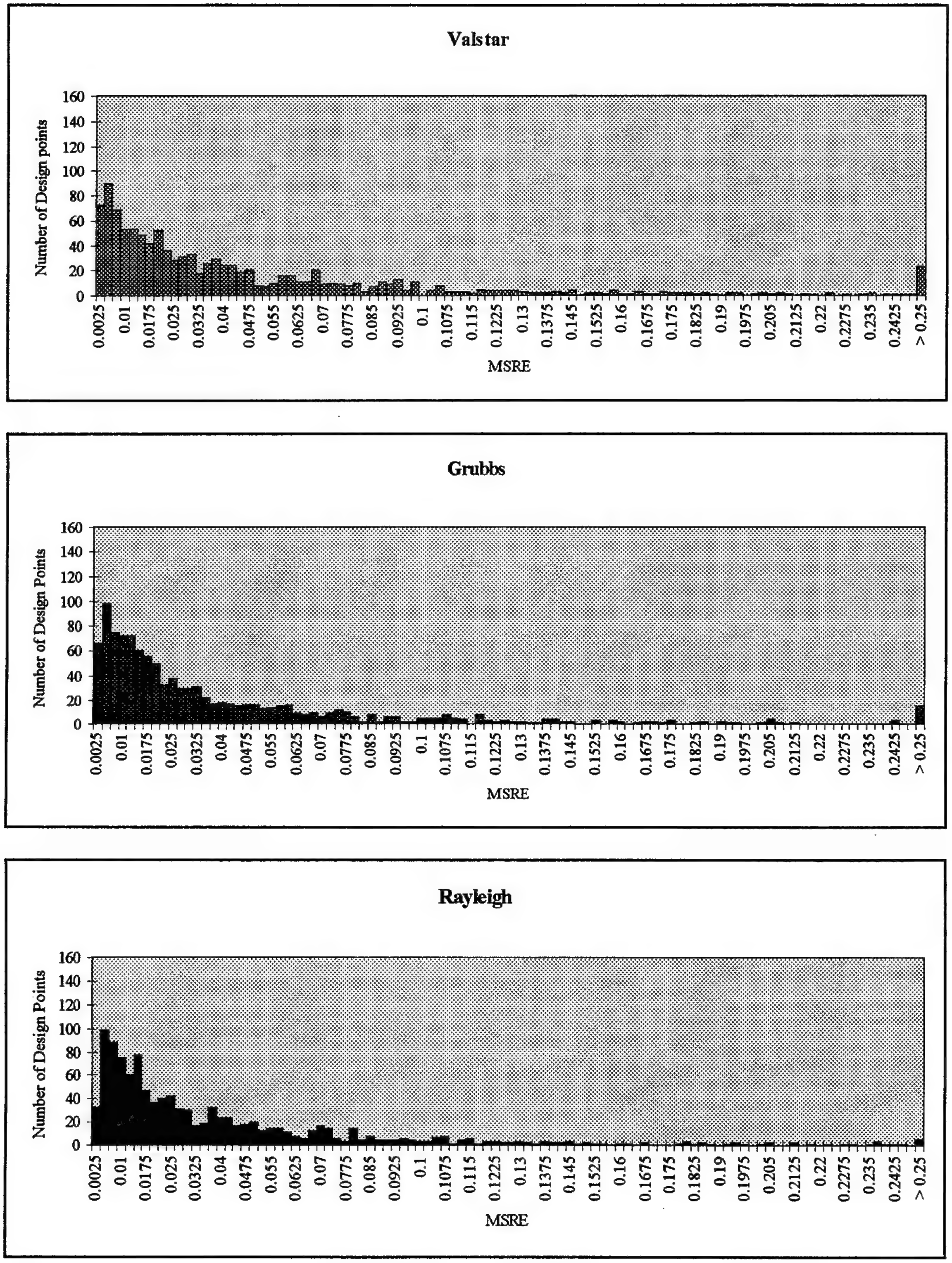
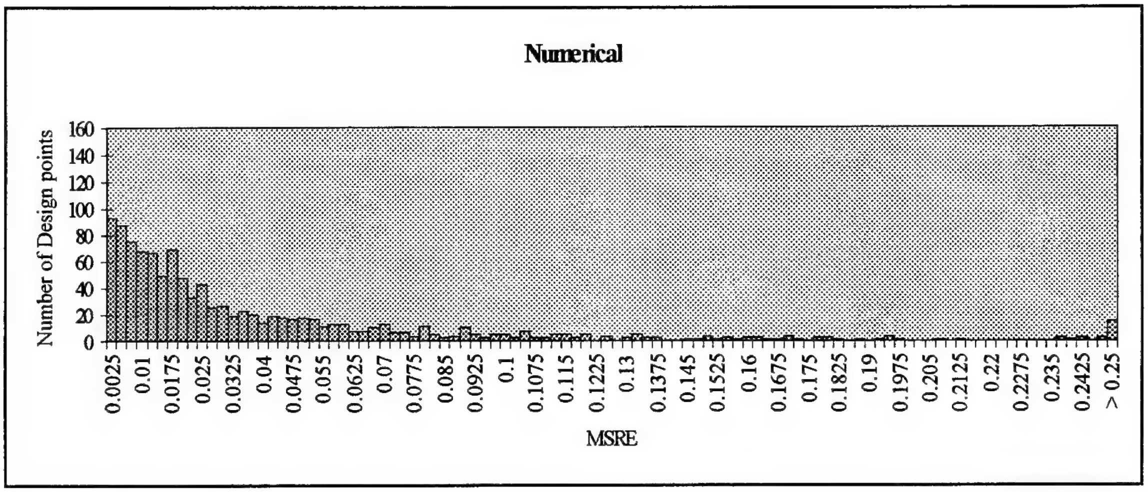


Figure 4.2 Overall Histograms Based on the Design Point Results (continued)



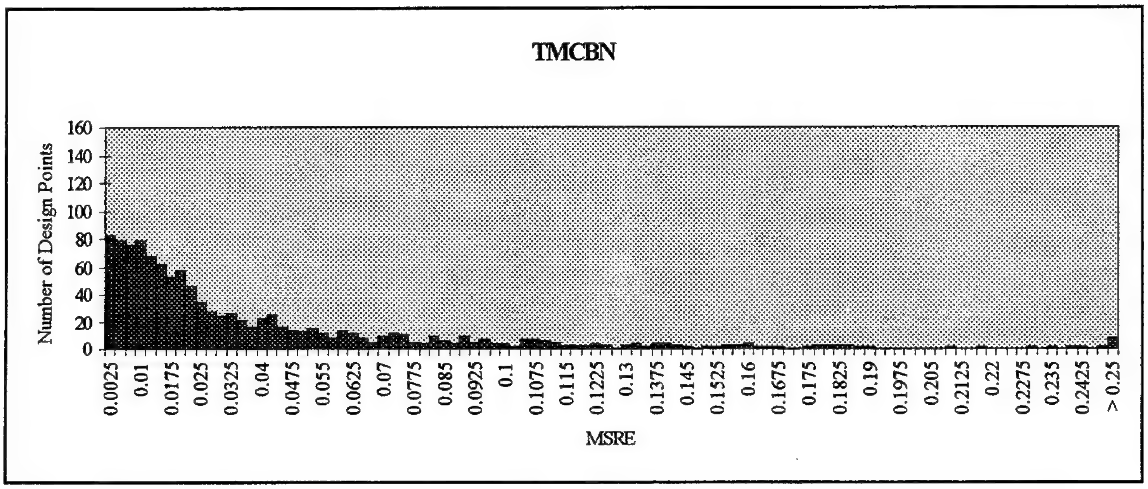


Figure 4.2 Overall Histograms Based on the Design Point Results (continued)

Considering these histograms, no single estimator stands out as being dominated by the other estimators. Each estimator except *MRand* shows a similar distribution of bars descending left to right. These histograms do, however, indicate weak performance from *Smed* and *MRand* based on the relatively larger number of design points which had MSRE values in excess of 0.25. The histogram for *MRand* is not surprising; as alluded to in Section 3.6, we calculated the *MRand* estimate for *all* design points, even those outside the estimator's validity bounds.

We next present the average MSRE values, the corresponding approximation for the MRE based on these MSRE averages, and the AE for each estimator below (bear in mind that the actual CEP was 100):

Statistic \ Estimator	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
Average MSRE	0.0564	0.0398	1.6525	0.0478	0.0391	0.0391	0.0421	0.0399
Corresponding MRE Approximation	0.2376	0.1995	1.2855	0.2187	0.1977	0.1976	0.2051	0.1997
AE (True CEP was 100)	±23.758	±19.951	±128.55	±21.873	±19.769	±19.764	±20.507	±19.969

Table 4.3 Overall Design Point Statistics

MRand, Smed, and *Valstar* appear relatively weak in Table 4.3, since they have the highest overall error statistics. CDF plots based on the overall design point results further indicate relatively weak performance by *MRand, Smed,* and *Valstar*. The first plot in Figure 4.3 shows that *TMCBN* is stochastically dominated by *MRand, Smed,* and *Valstar*, while the second shows the mixed results of *TMCBN* and the other four CEP estimators under consideration:

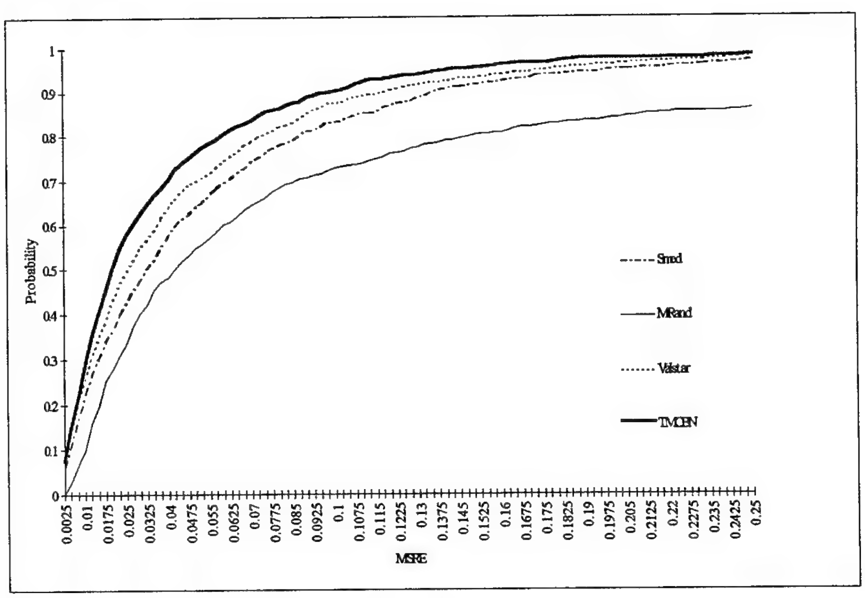


Figure 4.3 Overall CDF Plots Based on Our Design Point Results

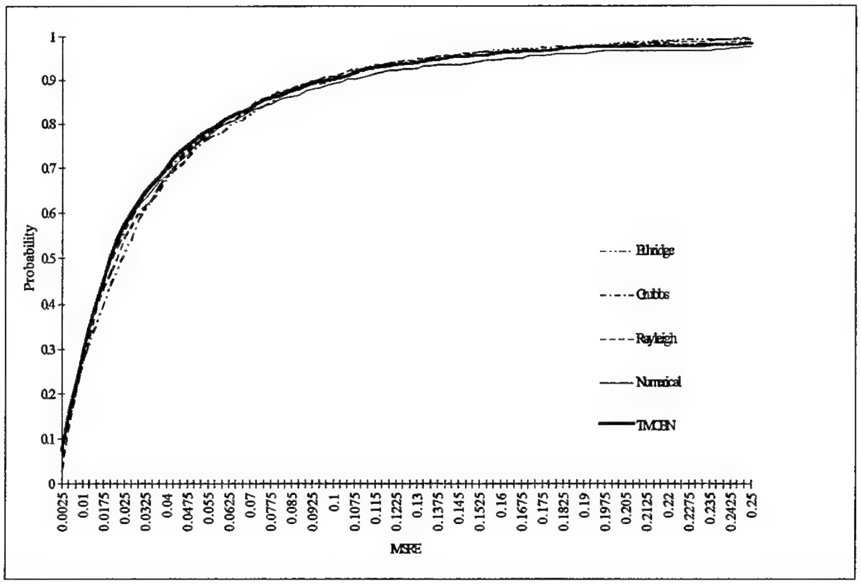


Figure 4.3 Overall CDF Plots Based on Our Design Point Results (continued)

As previously mentioned, it was clear that virtually all design points with a bias of 4.0σ were not valid for *MRand*. To try to clarify *MRand*'s performance, we next constructed a CDF plot for all of the design points *except* those with bias 4.0σ:

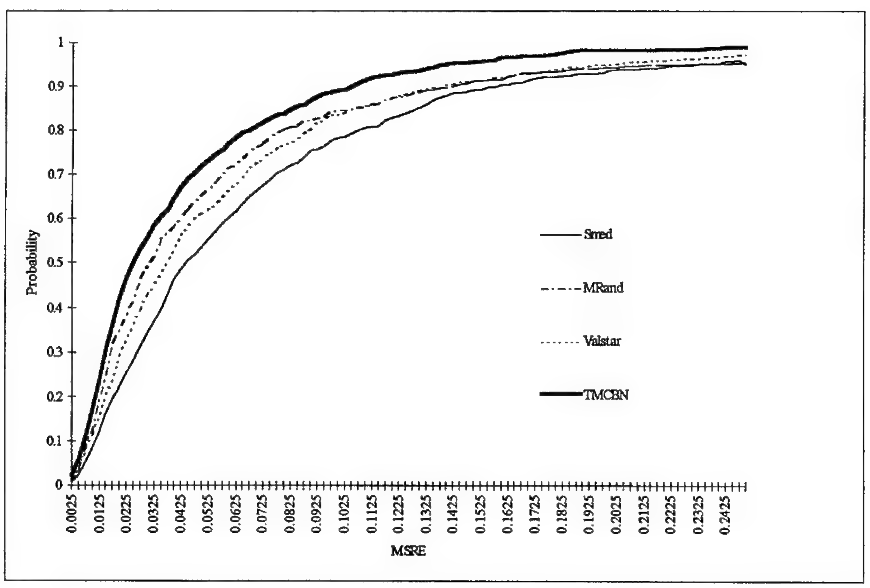


Figure 4.4 Overall CDF Plot for All Design Points Except Those With Bias 4.0σ

The CDF plot in Figure 4.4 still shows the same three estimators (including *MRand*) stochastically dominating *TMCBN*. The results of this plot indicate that *MRand* 's relatively weak performance is not entirely driven by design points with bias greater than 4.0σ.

To determine which factors appeared most significant, we produced these correlation tables for our design point results using Statistix software [1985]:

CORRELATIONS (PEARSON)

	N	BIAS	P	SD_RATIO	THETA	SMED	ETHRIDGE	MRAND	VALSTAR	GRUBBS
BIAS	-0.0000									
P	0.0000	-0.0400								
SD_RATIO	0.0000	-0.0000	0.0000							
THETA	-0.0000	0.1027	-0.2772	-0.0000						
SMED	-0.4425	-0.4521	0.1465	0.0219	-0.0268					
ETHRIDGE	-0.4794	-0.4584	0.1705	0.0088	-0.0420	0.8644				
MRAND	-0.0856	0.1144	0.0383	0.0477	0.0348	-0.0188	-0.0031			
VALSTAR	-0.4525	-0.4152	0.1707	0.0893	0.0063	0.8571	0.8667	-0.0119		
GRUBBS	-0.4862	-0.4120	0.1496	0.0763	0.0017	0.8694	0.8963	-0.0037	0.9854	
RAYLEIGH	-0.4454	-0.4629	0.1589	0.0640	-0.0173	0.8957	0.9340	-0.0079	0.9515	0.9645

CASES INCLUDED 1100 MISSING CASES 0

CORRELATIONS (PEARSON)

	N	BIAS	P	SD_RATIO	THETA	NUMERICAL
BIAS	0.0176					
P	-0.0027	-0.0399				
SD_RATIO	0.0032	-0.0074	-0.0030			
THETA	0.0026	0.1028	-0.2809	0.0056		
NUMERICAL	-0.3580	-0.2018	0.0826	0.1197	0.0214	
TMCBN	-0.3135	-0.1204	0.0643	0.1129	0.0315	0.9710

CASES INCLUDED 1088 MISSING CASES 12

Table 4.4 Statistix Correlation Tables for the Design Point Results

When examining these results, note that the correlation between our factors is not always zero. Because of the symmetries explained in Section 3.1 we used only positive correlation levels along the X-axis (where θ = 0); therefore bias and correlation appears correlated to θ in the upper table. In the lower table, the twelve design points that would

not converge for *Numerical* and *TMCBN* were not considered and account for the nonzero correlations between factors.

The correlation between either sample size or bias and any of the eight CEP estimators tends to be around -0.4, while the correlation between any of the remaining factors and any of the eight CEP estimators falls into a range of [-0.0485, 0.1707]. From these results, sample size and bias are clearly the factors most strongly influencing the MSRE scores. This judgment corresponds to that of Tongue, who also independently determined sample size and bias to be the most crucial factors [Tongue (1993), page 5-7].

Having resolved sample size and bias to be the most significant factors, we next analyzed the design points for each specific sample size and bias factor level. Although each case was analyzed, we group cases that produced identical results together for presentation. Based on their weak showing in the overall CDFs and MSRE averages presented earlier, we did not consider *MRand, Smed,* or *Valstar* in this casewise analysis.

We first present results for our sample size cases, beginning with error statistics. We received essentially identical results for sample sizes fifteen, nine, and six; these cases are grouped into a single set for presentation:

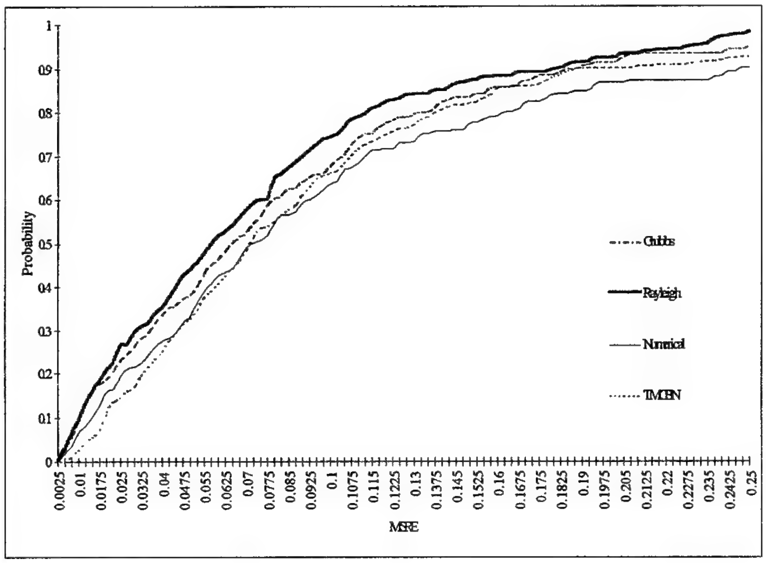
Sample Size = 15, 9, or 6:	*Ethridge*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
Average MSRE	0.0269	0.024	0.0269	0.023	0.0212
Corresponding MRE Approximation	0.1639	0.1548	0.1639	0.1518	0.1458
AE (True CEP was 100)	±16.386	±15.483	±16.392	±15.177	±14.575

Sample Size = 3:	*Ethridge*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
Average MSRE	0.0787	0.0844	0.0756	0.1017	0.0983
Corresponding MRE Approximation	0.2805	0.2905	0.275	0.3189	0.3136
AE (True CEP was 100)	±28.047	±29.054	±27.501	±31.892	±31.356

Table 4.5 Statistics for Design Point Sample Size Cases

The statistics for our two sample size cases indicate that based on our MSRE results the five considered CEP estimators produce errors of approximately 14-17% for sample sizes fifteen, nine, and six; this rate jumps to around 27-32% at sample size three.

We next present a CDF plot for each of these sample size sets:



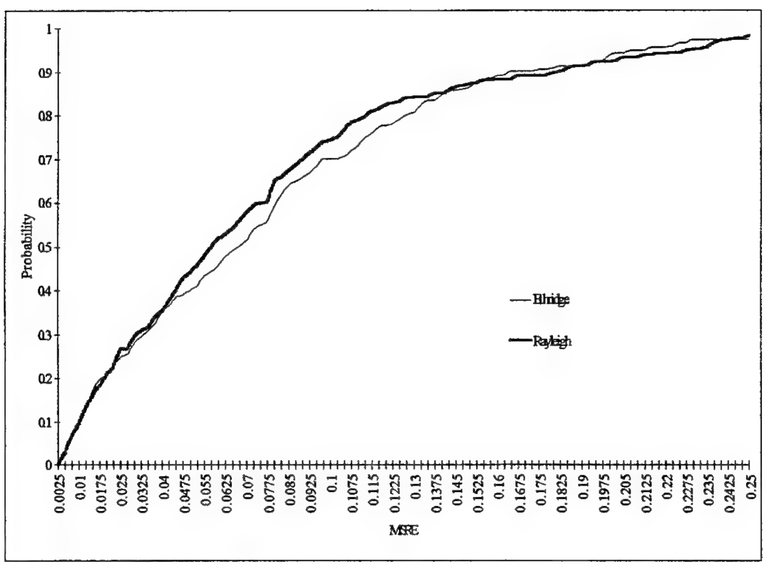


Figure 4.5 CDF Plots Based on Our Design Point Results (n = 3)

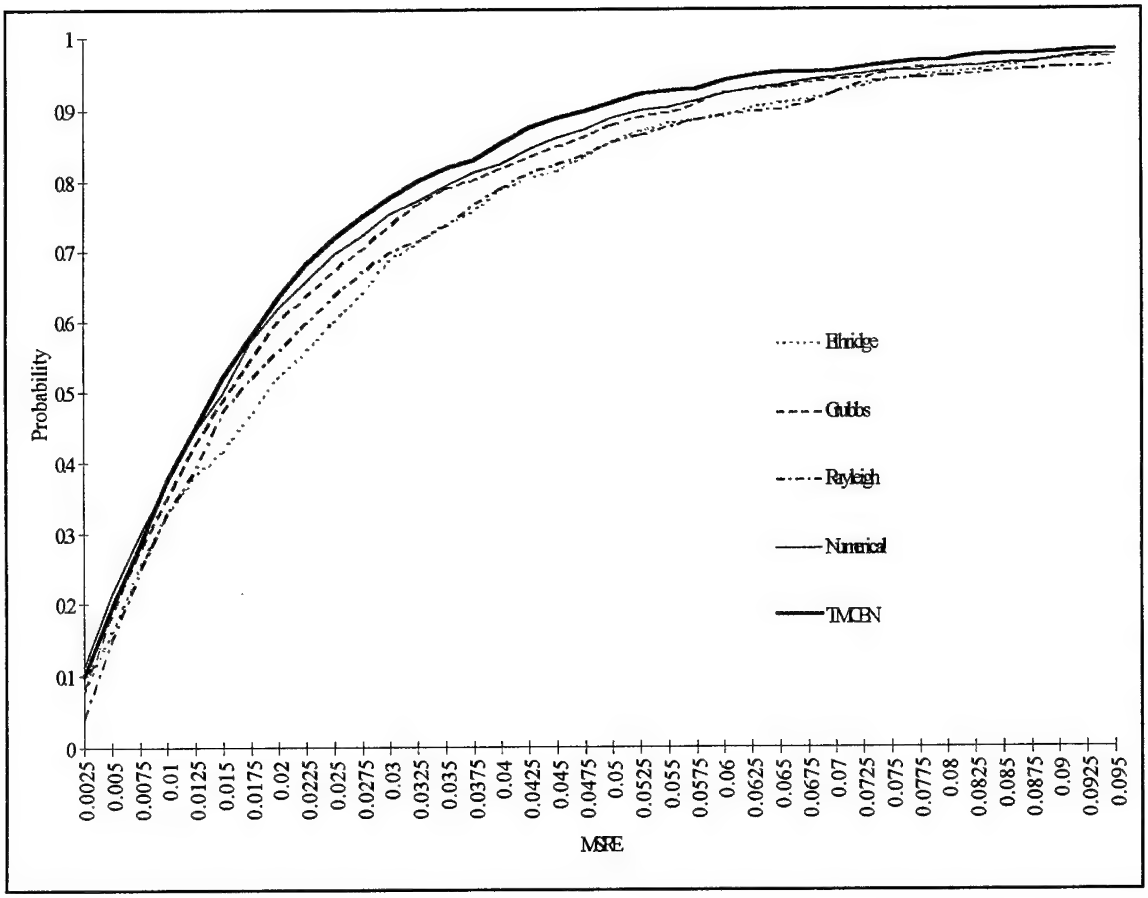


Figure 4.6 CDF Plot Based on Our Design Point Results (n = 15, 9, or 6)

As seen in the CDF plots in Figures 4.5 and 4.6, *TMCBN* appears best for sample sizes fifteen, nine, and six, while *Rayleigh and Ethridge* are stochastically dominated at sample size three. Bear in mind when observing these plots that a stochastically dominated distribution indicates *good* performance, since high MSRE is undesired.

We next examine the design point results classified according to bias. The design points with bias 0, 0.5σ, or 1.0σ yielded similar results and are grouped together for presentation. Like the sample size cases, we first present statistics related to the design points classified by bias followed by CDF plots.

Bias = 0, 0.5σ, 1.0σ	*Ethridge*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
Average MSRE	0.0581	0.0574	0.0588	0.0575	0.0507
Corresponding MRE Approximation	0.241	0.2396	0.2424	0.2398	0.2251
AE (True CEP was 100)	±24.095	±23.957	±24.24	±23.984	±22.508

Bias = 2.0σ	*Ethridge*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
Average MSRE	0.0295	0.0287	0.0246	0.0288	0.0283
Corresponding MRE Approximation	0.1717	0.1695	0.1568	0.1697	0.1681
AE (True CEP was 100)	±17.175	±16.949	±15.684	±16.969	±16.808

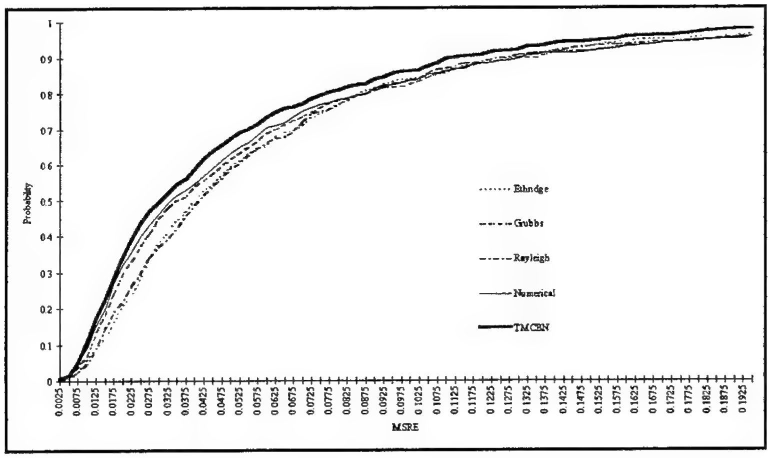
Bias = 4.0σ	*Ethridge*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
Average MSRE	0.0094	0.0086	0.0096	0.02	0.0269
Corresponding MRE Approximation	0.0969	0.0927	0.0979	0.1413	0.1641
AE (True CEP was 100)	±9.6902	±9.2676	±9.7858	±14.127	±16.414

Table 4.6 Statistics for Design Point Bias Cases

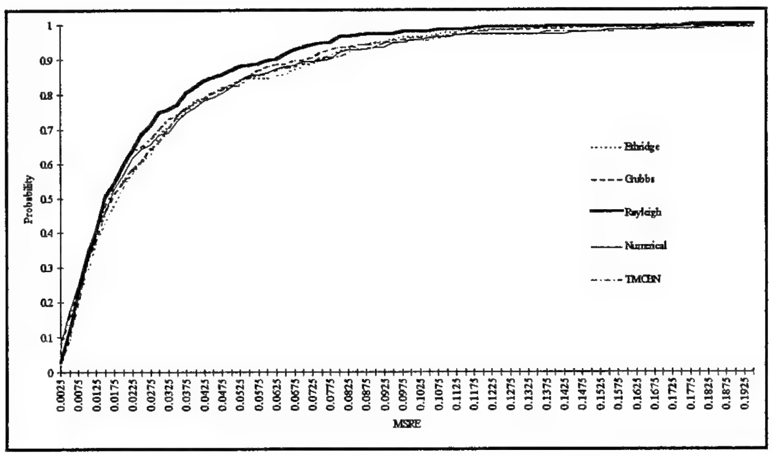
From these bias statistics, it is clear that the margin of difference in the expected error is relatively small for design points where bias was less than 4.0σ. The expected error ranges from around 22-24% for bias (0, 0.5σ, 1.0σ) and around 15-17% for design points with bias = 2.0σ. The MSRE average values (and thus the expected error) of the two numerical estimators *Numerical* and *TMCBN*, however, appear substantially higher at bias 4.0σ than the other three considered estimators.

We next constructed CDF plots for each of these three bias cases. These plots are displayed in Figure 4.7:

Design Points With Bias 0, 0.5σ, or 1.0σ



Design Points With Bias = 2.0σ:



Design Points With Bias = 4.0σ:

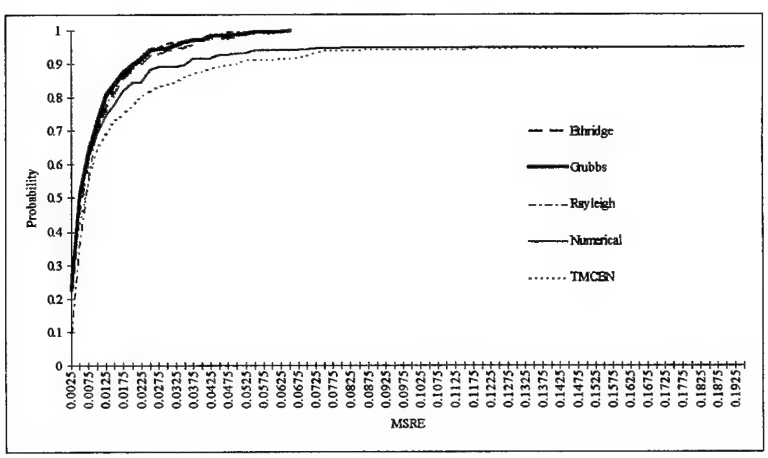


Figure 4.7 CDF Plots Based on Design Point Bias Cases

According to the CDF plots shown in Figures 4.7 and 4.8, *TMCBN* appeared to perform best for bias levels of 1.0σ and under while *Rayleigh* appeared to perform best for bias 2.0σ. At bias 4.0σ, *Ethridge*, *Grubbs,* and *Rayleigh* appeared to outperform the two numerical estimators.

While our CDF plots based on sample size or bias cases give an indication of the strengths and weaknesses of the considered CEP estimators, there is conflicting "overlap" for specific cases. For example, *TMCBN* appears best for sample size fifteen and *Rayleigh* appears best for bias 2.0σ - which is best for the specific case of sample size 15 and bias 2.0σ? To allow a comparison of the estimators for specific sample size and bias combinations, we next present tables that display for each estimator the MSRE and AE results for each sample size/bias case:

Table 4.7 Average MSRE Results for Design Point Sample Size/Bias Combinations

n	bias	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
15	0	0.0444	0.0259	0.0145	0.0177	0.017	0.0268	0.0167	0.0163
15	0.5σ	0.0382	0.023	0.0177	0.0229	0.0175	0.0266	0.0168	0.0161
15	1.0σ	0.0245	0.0178	0.0178	0.0228	0.0146	0.0176	0.0131	0.0127
15	2.0σ	0.0112	0.0115	0.0101	0.0123	0.0087	0.0079	0.008	0.008
15	4.0σ	0.0035	0.0033	0.0834	0.0027	0.0025	0.0045	0.0023	0.0023
9	0	0.0597	0.0455	0.0286	0.0352	0.0316	0.0462	0.0311	0.0268
9	0.5σ	0.0629	0.0379	0.0354	0.0445	0.0346	0.0429	0.0334	0.0301
9	1.0σ	0.0464	0.0308	0.0341	0.042	0.0295	0.0302	0.0277	0.0252
9	2.0σ	0.0272	0.019	0.0177	0.0223	0.0171	0.015	0.0166	0.0163
9	4.0σ	0.0065	0.0061	0.1556	0.0053	0.0052	0.0068	0.005	0.0056
6	0	0.1135	0.087	0.0664	0.0839	0.069	0.0881	0.068	0.0573
6	0.5σ	0.0929	0.0693	0.0631	0.0779	0.0645	0.0731	0.0627	0.0531
6	1.0σ	0.059	0.046	0.049	0.0591	0.0447	0.0436	0.0421	0.0385
6	2.0σ	0.0385	0.0299	0.0314	0.0366	0.0291	0.0256	0.0288	0.0281
6	4.0σ	0.0109	0.0096	0.7489	0.0094	0.0091	0.01	0.0091	0.0105
3	0	0.2186	0.1408	0.1458	0.1841	0.1584	0.1408	0.1564	0.1334
3	0.5σ	0.1945	0.1255	0.1517	0.1641	0.1402	0.128	0.152	0.1302
3	1.0σ	0.1326	0.0988	0.168	0.1239	0.1029	0.0926	0.1034	0.0927
3	2.0σ	0.0816	0.0576	0.9567	0.0716	0.06	0.0499	0.0623	0.0612
3	4.0σ	0.0241	0.0185	25.366	0.0182	0.0175	0.017	0.0712	0.1008

Table 4.8 Approximate Estimation Error (AE) Based on Design Point Average MSRE Results for Each Sample Size/Bias Case (Actual CEP was 100)

n	bias	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
15	0	±21.071	±16.093	±12.042	±13.304	±13.038	±16.371	±12.923	±12.767
15	0.5σ	±19.545	±15.166	±13.304	±15.133	±13.229	±16.31	±12.961	±12.689
15	1.0σ	±15.652	±13.342	±13.342	±15.1	±12.083	±13.266	±11.446	±11.269
15	2.0σ	±10.583	±10.724	±10.05	±11.091	±9.3274	±8.8882	±8.9443	±8.9443
15	4.0σ	±5.9161	±5.7446	±28.879	±5.1962	±5	±6.7082	±4.7958	±4.7958
9	0	±24.434	±21.331	±16.912	±18.762	±17.776	±21.494	±17.635	±16.371
9	0.5σ	±25.08	±19.468	±18.815	±21.095	±18.601	±20.712	±18.276	±17.349
9	1.0σ	±21.541	±17.55	±18.466	±20.494	±17.176	±17.378	±16.643	±15.875
9	2.0σ	±16.492	±13.784	±13.304	±14.933	±13.077	±12.247	±12.884	±12.767
9	4.0σ	±8.0623	±7.8102	±39.446	±7.2801	±7.2111	±8.2462	±7.0711	±7.4833
6	0	±33.69	±29.496	±25.768	±28.965	±26.268	±29.682	±26.077	±23.937
6	0.5σ	±30.48	±26.325	±25.12	±27.911	±25.397	±27.037	±25.04	±23.043
6	1.0σ	±24.29	±21.448	±22.136	±24.31	±21.142	±20.881	±20.518	±19.621
6	2.0σ	±19.621	±17.292	±17.72	±19.131	±17.059	±16	±16.971	±16.763
6	4.0σ	±10.44	±9.798	±86.539	±9.6954	±9.5394	±10	±9.5394	±10.247
3	0	±46.755	±37.523	±38.184	±42.907	±39.799	±37.523	±39.547	±36.524
3	0.5σ	±44.102	±35.426	±38.949	±40.509	±37.443	±35.777	±38.987	±36.083
3	1.0σ	±36.414	±31.432	±40.988	±35.199	±32.078	±30.43	±32.156	±30.447
3	2.0σ	±28.566	±24	±97.811	±26.758	±24.495	±22.338	±24.96	±24.739
3	4.0σ	±15.524	±13.601	±503.65	±13.491	±13.229	±13.038	±26.683	±31.749

Thus far in our design point presentation we have only presented MSRE results. In Figures 4.8 through 4.11, we display the MRE and VRE results which correspond to the MSRE values presented in Table 4.6. For each sample size and bias case, these plots show the MRE (labeled) along with a range of values plus or minus two standard deviations from the MRE. This range indicates the VRE scores for each case, since one standard deviation from the MRE is equal to the square root of the VRE.

In these plots, note that the range of values two standard deviations below the MRE extends past zero on the low side. These negative values are due to our assumption that the MRE scores for each estimator follow a normal distribution. Obviously we had no negative RE scores and these negative portions of the ranges can be ignored.

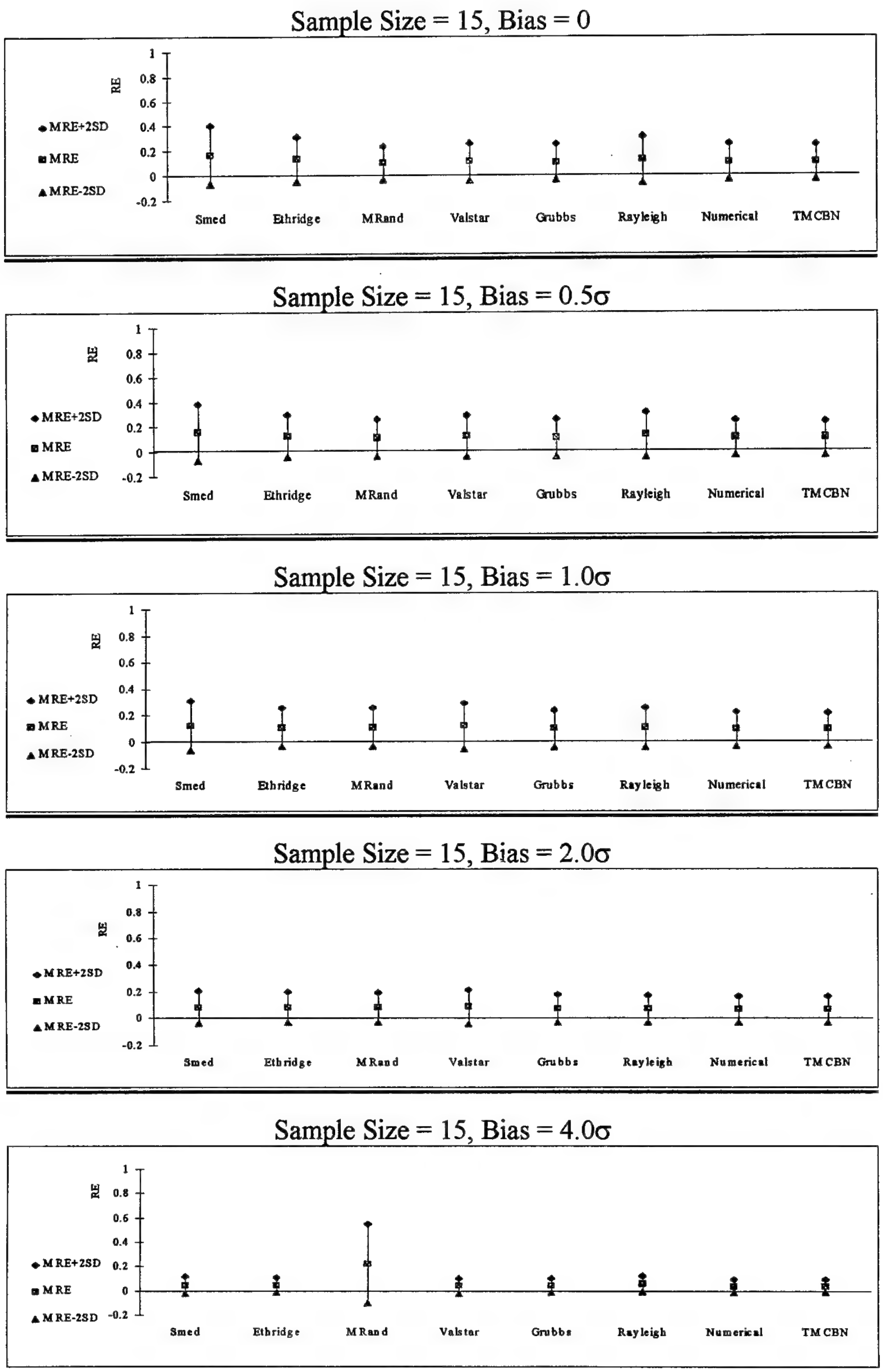


Figure 4.8 Design Point RE Plots for Each Bias Case for Sample Size Fifteen

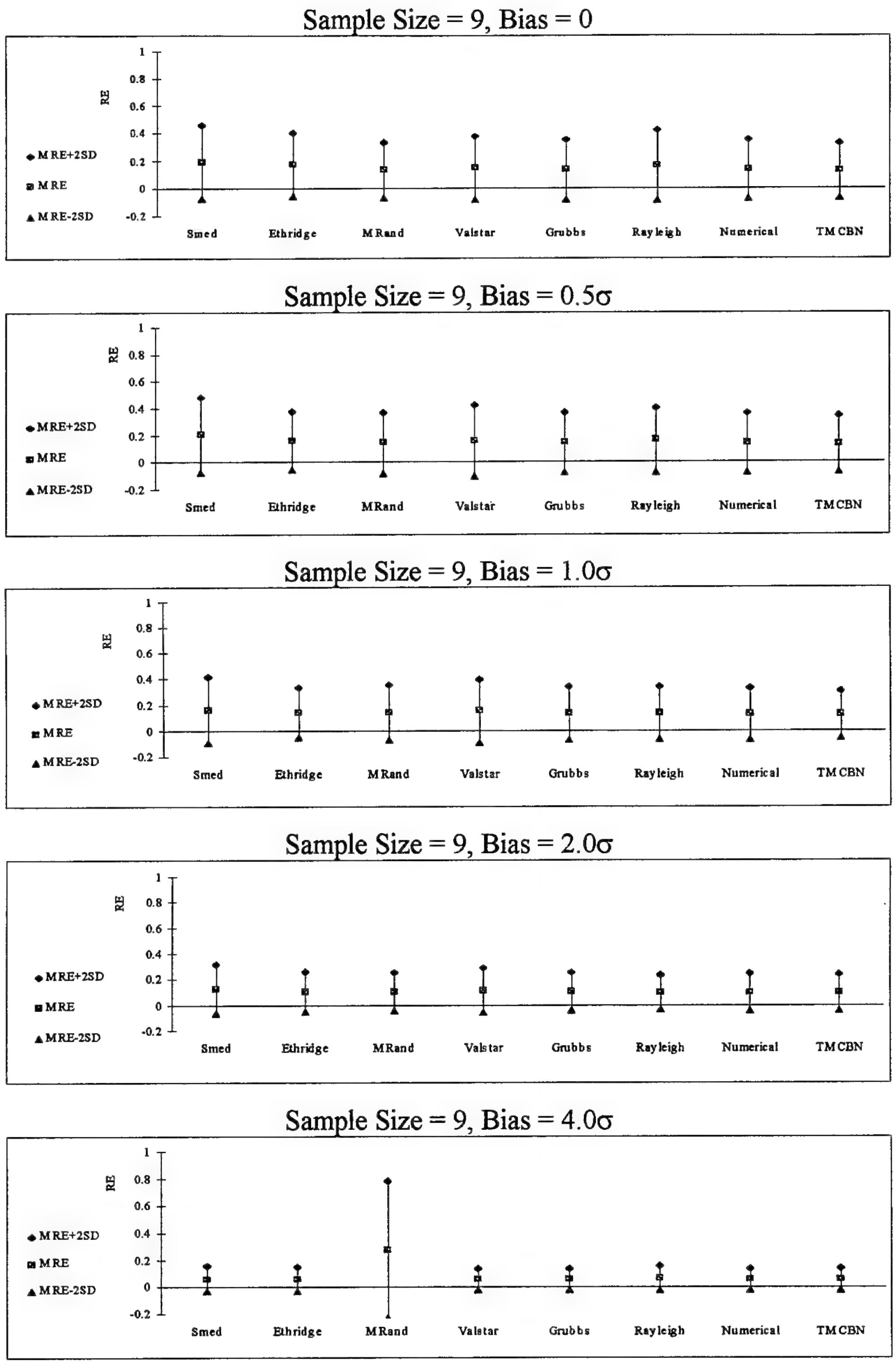
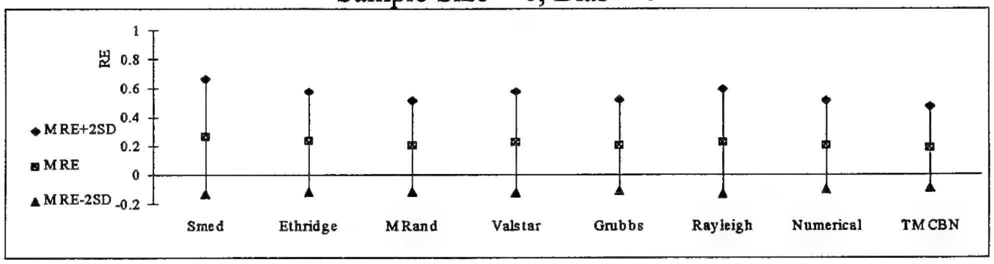
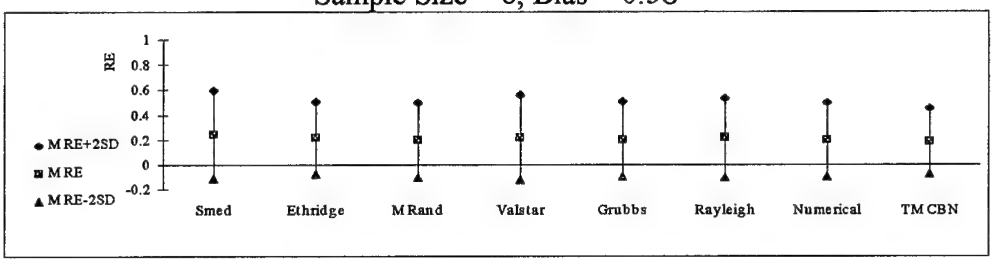


Figure 4.9 Design Point RE Plots for Each Bias Case for Sample Size Nine

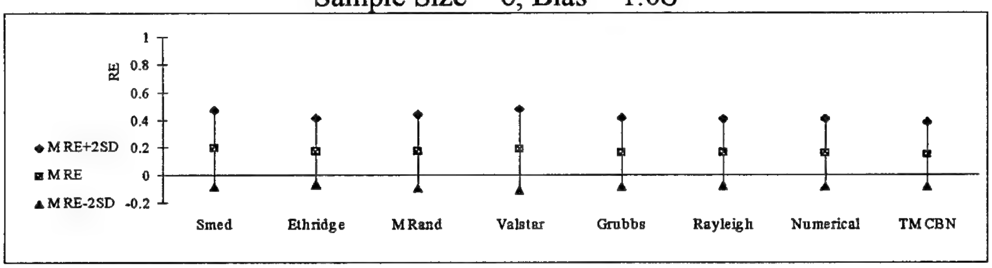
Sample Size = 6, Bias = 0



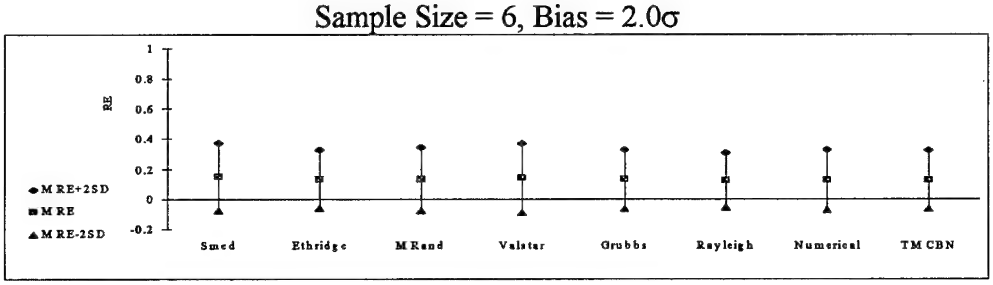
Sample Size = 6, Bias = 0.5σ



Sample Size = 6, Bias = 1.0σ



Sample Size = 6, Bias = 2.0σ



Sample Size = 6, Bias = 4.0σ

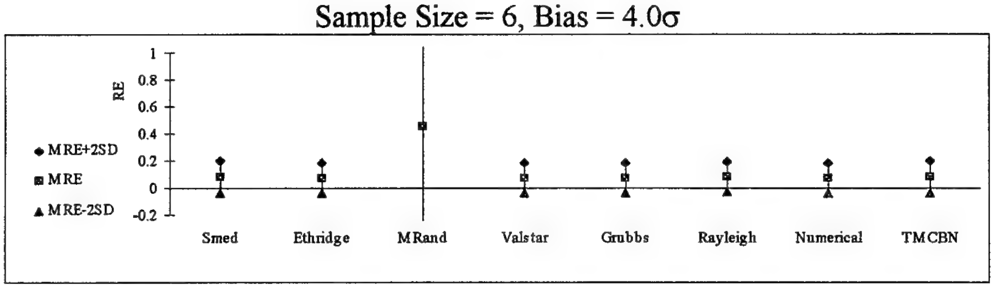


Figure 4.10 Design Point RE Plots for Each Bias Case for Sample Size Six

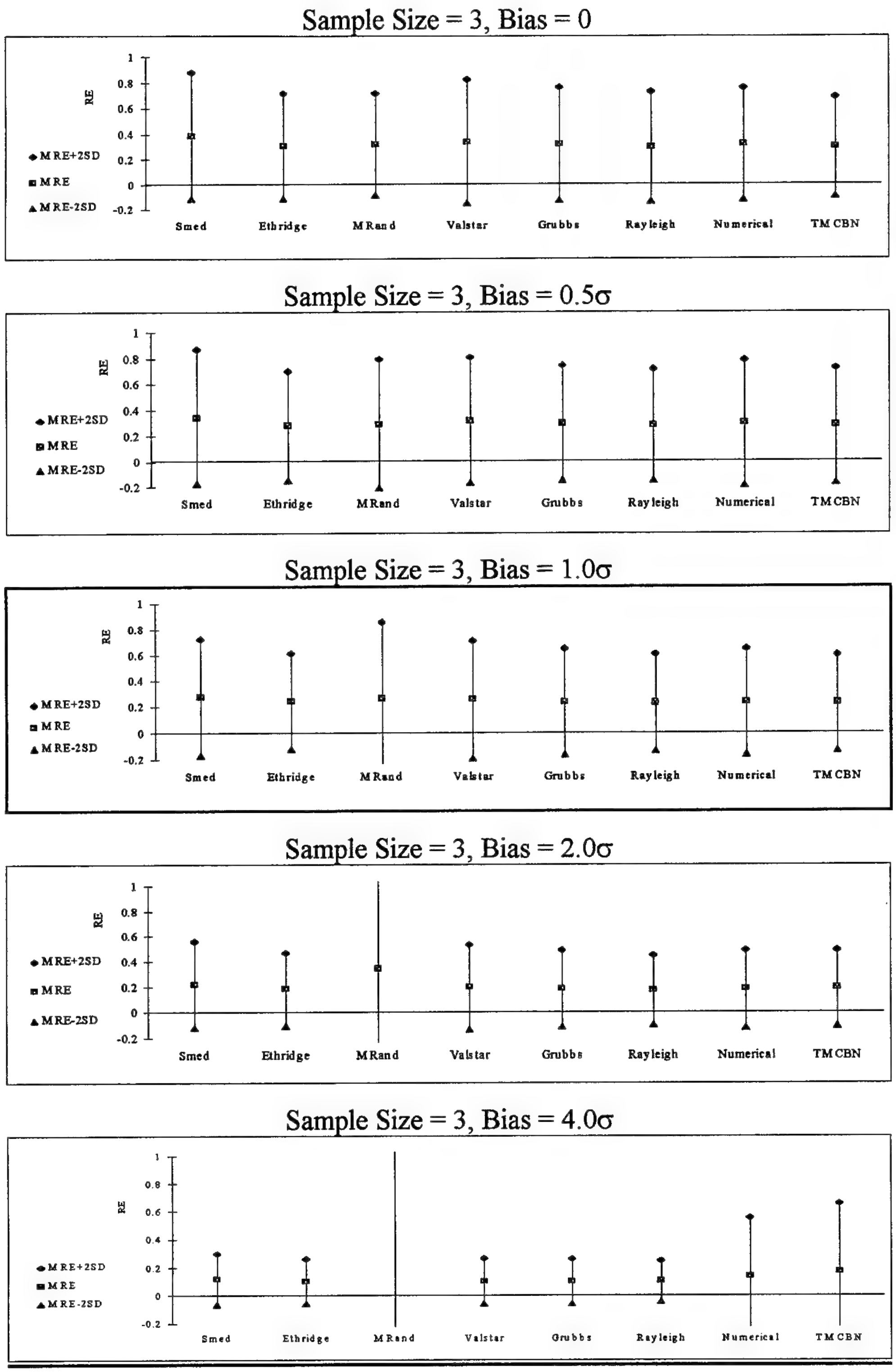


Figure 4.11 Design Point RE Plots for Each Bias Case for Sample Size Three

These plots indicate the following about our design point MRE and VRE results:

1. The difference in MRE and VRE was marginal for most cases. Only *MRand*'s poor scores at high bias cases clearly stood out, and this was primarily due to using the estimator for samples for which it was not valid.

2. Both MRE and VRE tend to increase at smaller sample sizes and smaller bias levels; conversely, these MOEs tend to decrease for larger sample sizes and larger bias levels.

4.3 Comparison With Previous Studies

We next compare our design point results with those from the previous CEP comparison studies of Elder (1986), Puhek (1992), and Tongue (1993).

1. Elder: Elder compared *MRand, Grubbs,* and the Grubbs-Patniak/ Wilson-Hilferty estimator in his simulation experiment. Based on the number of design points for each estimator that yielded the minimum RE, Elder concluded that *Grubbs* had equal or better results than *MRand* for most bias and ellipticity cases [Elder (1986), 4-2 through 4-4]. Our design point results indicate a similar conclusion:

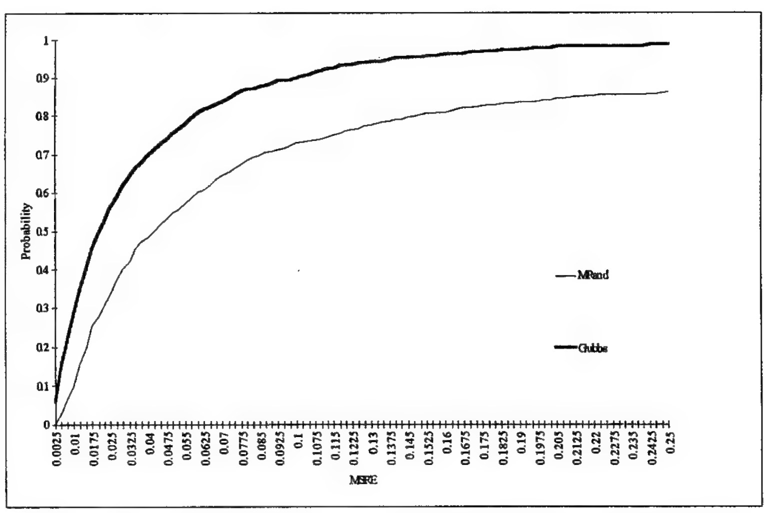


Figure 4.12 Our Overall Design Point Results for *Grubbs* and *MRand*

2. *Puhek*: Puhek (1992) considered *Grubbs, MRand, Ethridge,* and *Rayleigh* in his comparison study, concluding that *Rayleigh* generally dominated the comparison. He rated *Rayleigh* superior for all design points except those where bias equaled 1.5σ, where *Grubbs* and *Rayleigh* were judged to have equal performance [Puhek(1992), page 4-8].

Our results did not show such dominance by *Rayleigh*. According to our results for sample sizes fifteen, nine and six, *Grubbs* appears best for these four estimators, as indicated in the CDF plot below:

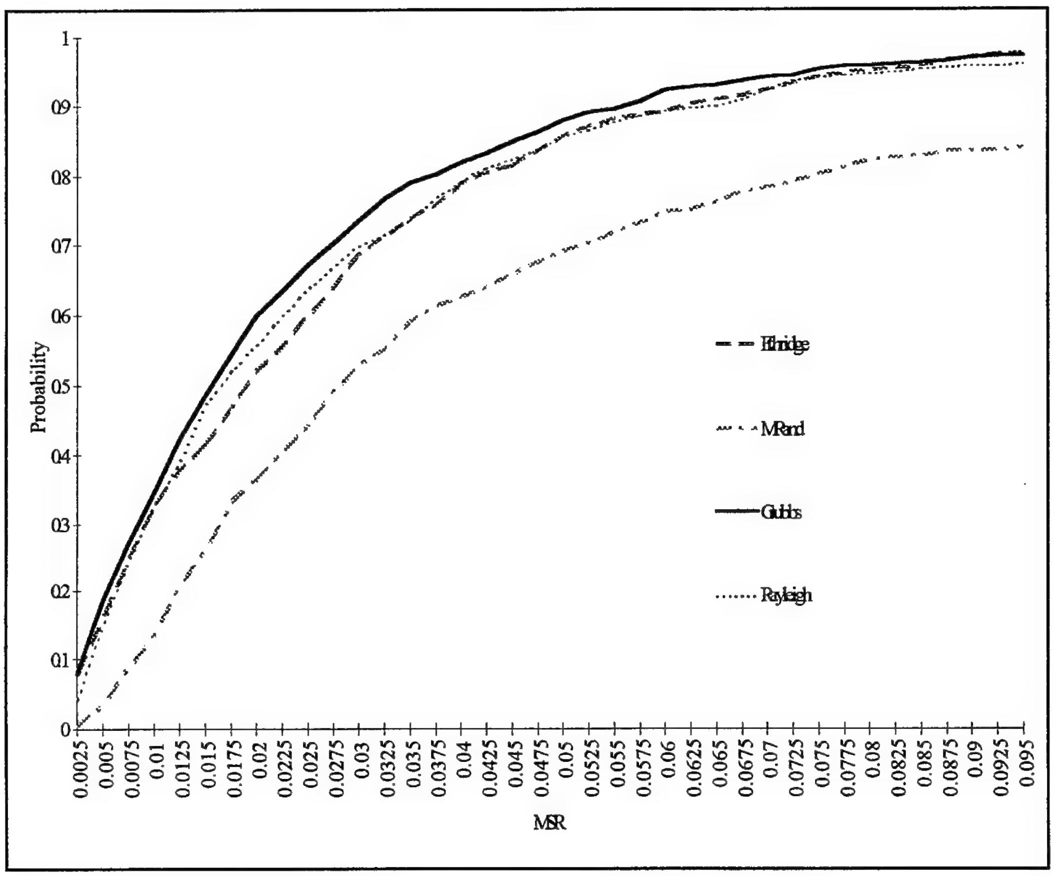


Figure 4.13 Our Design Point Results for the Estimators Considered by Puhek at Sample Sizes Fifteen, Nine, and Six

We can only speculate why our results were much different from Puhek's. One

possible explanation is the fact that our experimental designs were different. Puhek used only positive correlation levels and the largest bias he considered was 2σ. In addition, Puhek's conclusions were based strictly on MRE. Even considering only the equivalent portion of our results leads to different conclusions, however.

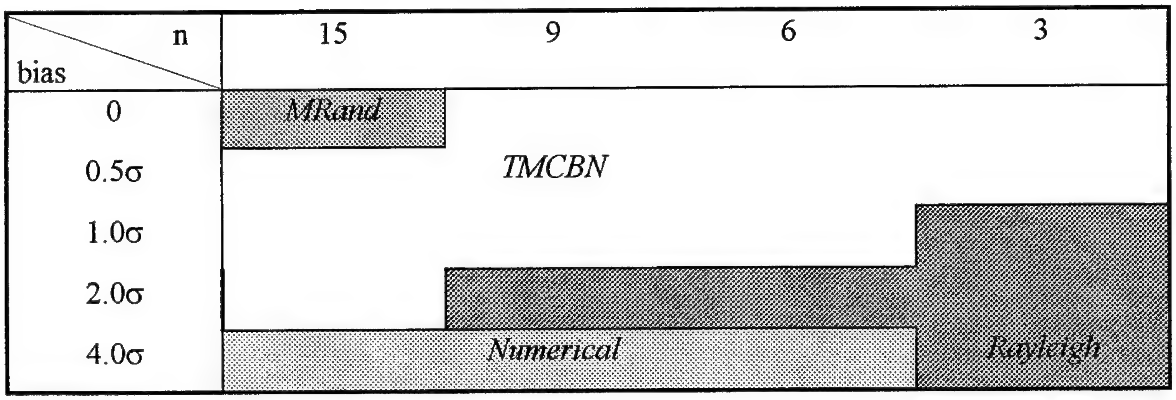
Another possible reason for this disparity is the fact that at each design point Puhek only had *one* relative error estimate for each of the four CEP estimators under consideration. With the replications used in our experiment, each design point has an *average* relative error estimate. The potential for variance was thus much more prevalent in Puhek's study and may have led to misleading results.

3. Tongue: Tongue's study consisted of two comparison experiments. In the first, he compared the four estimators from Puhek's study along with *CBN*, a numerical integration CEP estimator discussed in Section 2.3. He then repeated the experiment, replacing *CBN* with his empirically derived *TMCBN* estimator. It is this second experiment that we compare our overall design point results to.

Tongue's results are quite similar to ours. As previously mentioned, he determined sample size and bias to be the two most significant factors in his simulation experiment, in agreement with our results. For most sample size and bias combinations, Tongue concluded that his *TMCBN* estimator performed best [Tongue (1993), page 6-13].

Tongue constructed a sample size/bias decision grid based on MRE [Tongue(1993), page 6-13]. Comparing Tongue's grid to our grid which displays the estimator with the minimum average MSRE for each sample size/bias case demonstrates the similarity in results:

Our Grid Showing the CEP Estimator With the Minimum Average MSRE, Based on Our Design Point Results



Tongue's Overall Decision Grid Based on his MRE Results

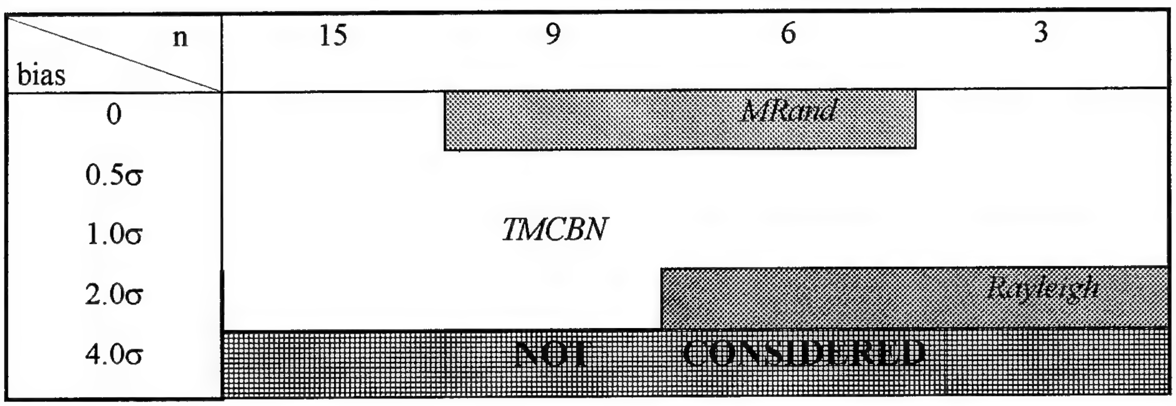


Figure 4.14 Our Design Point Grid Contrasted Against Tongue's Decision Grid

4.4 The Sample Analysis Set Results

In this section, we repeat the presentation sequence used earlier in Section 4.2 for the sample analysis set results. Before we proceed, we first remind the reader how the sample analysis sets were constructed. When we use the word "sample" in this thesis, we refer to a set of either three, six, nine, or fifteen (x,y) coordinates, depending on the sample size. At each of our 1,100 design points, we created ten simulated samples. The union of these 11,000 samples is identical to the union of the 11,000 samples that are

elements of some sample analysis set. The samples are just classified differently in each case. The sample elements for each design point were classified according to the population parameters used to generate the samples, while the sample analysis sets are categorized according to the sample statistics of each sample.

Just as we did for the design point results, we begin our presentation of the sample analysis set results by displaying the overall histograms for each CEP estimator considered:

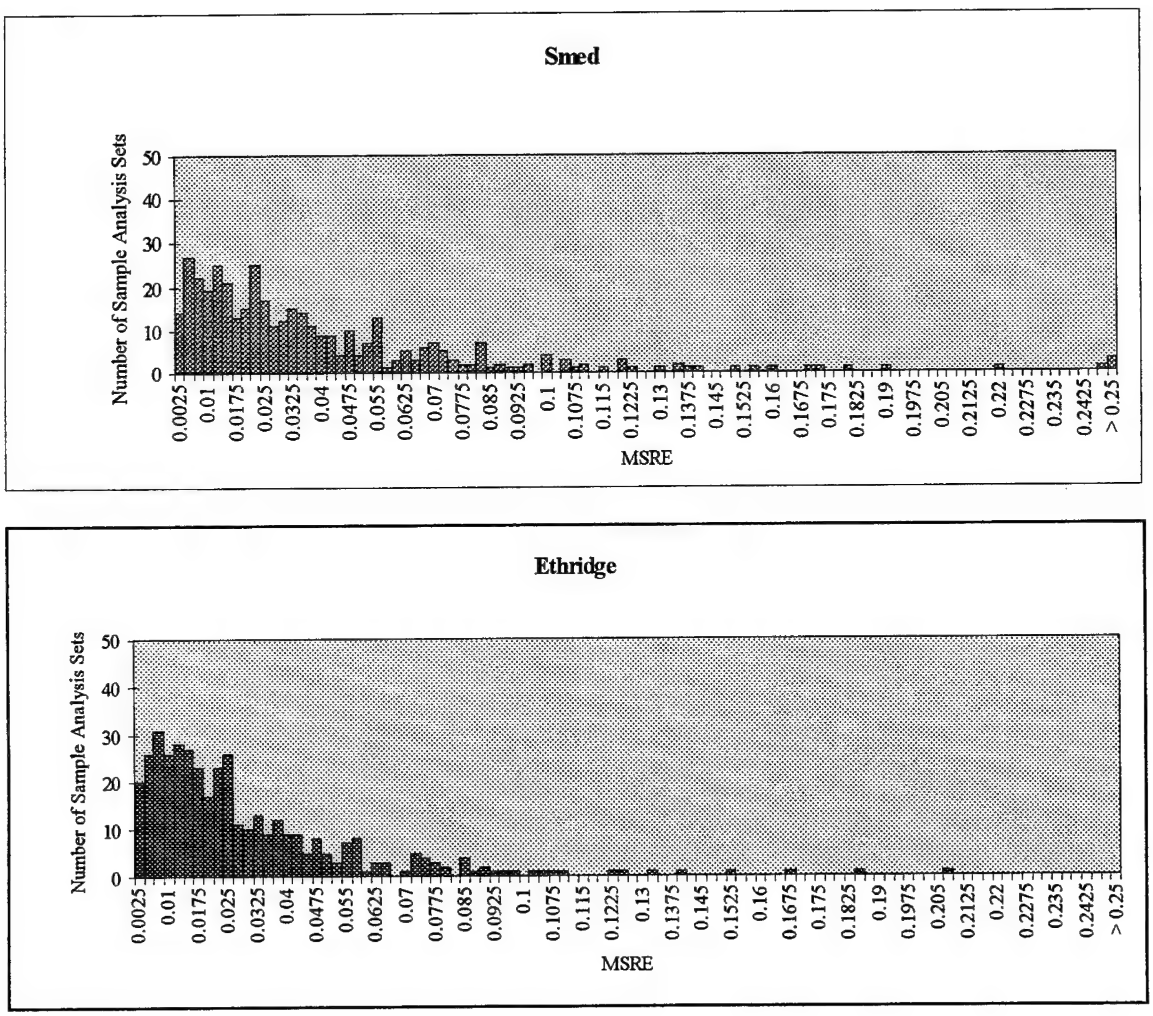
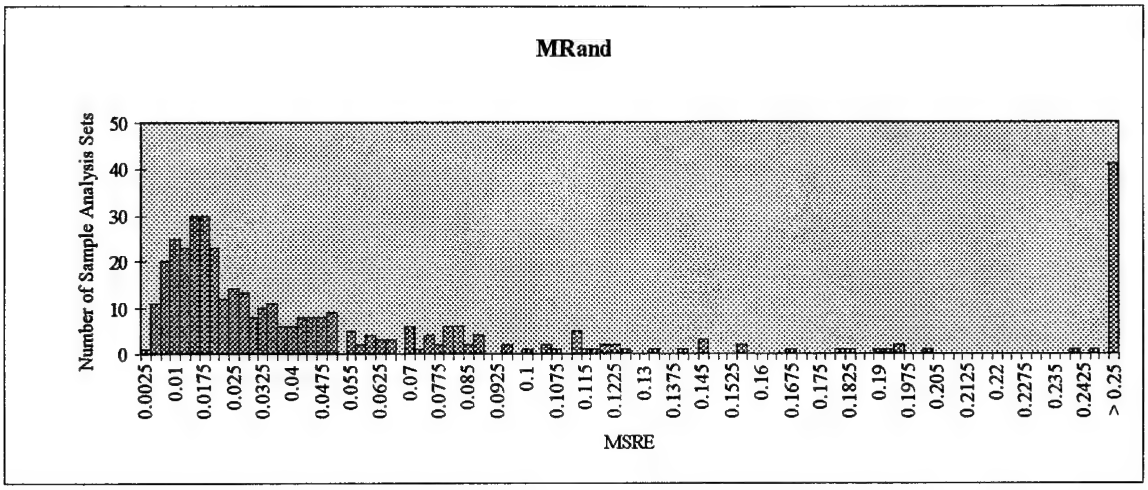
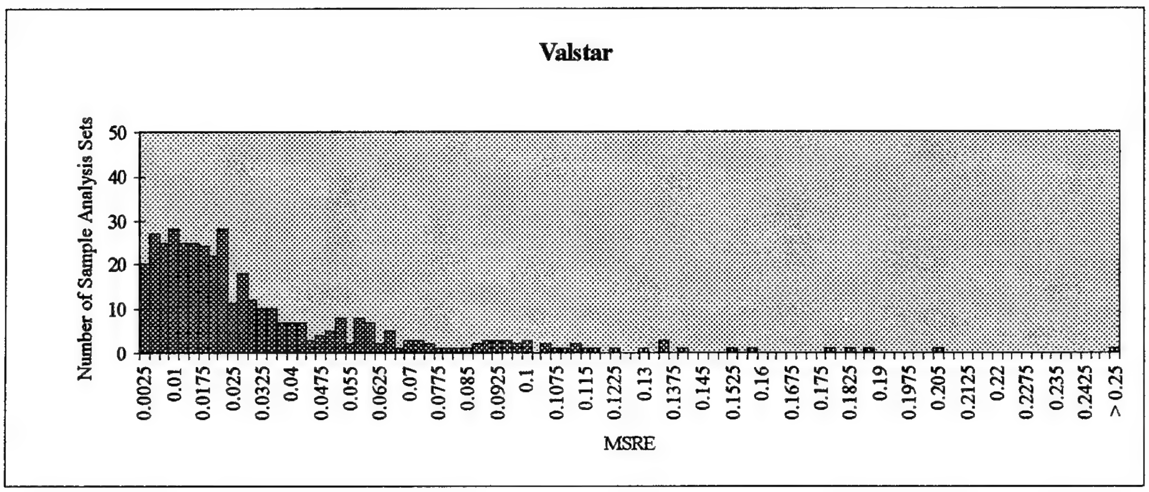


Figure 4.15 Overall Histograms Based on the Sample Analysis Set Results





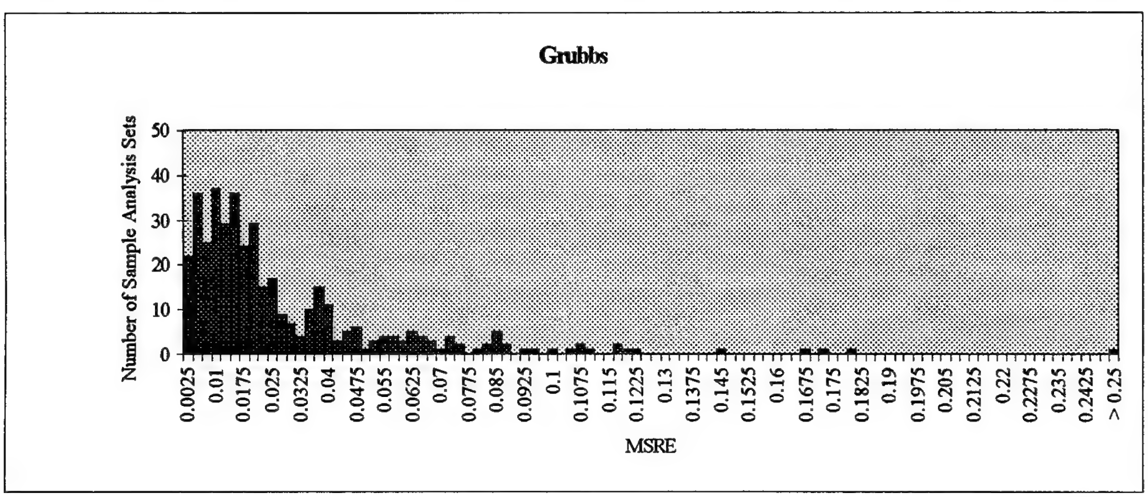


Figure 4.15 Overall Histograms Based on the Sample Analysis Set Results (continued)

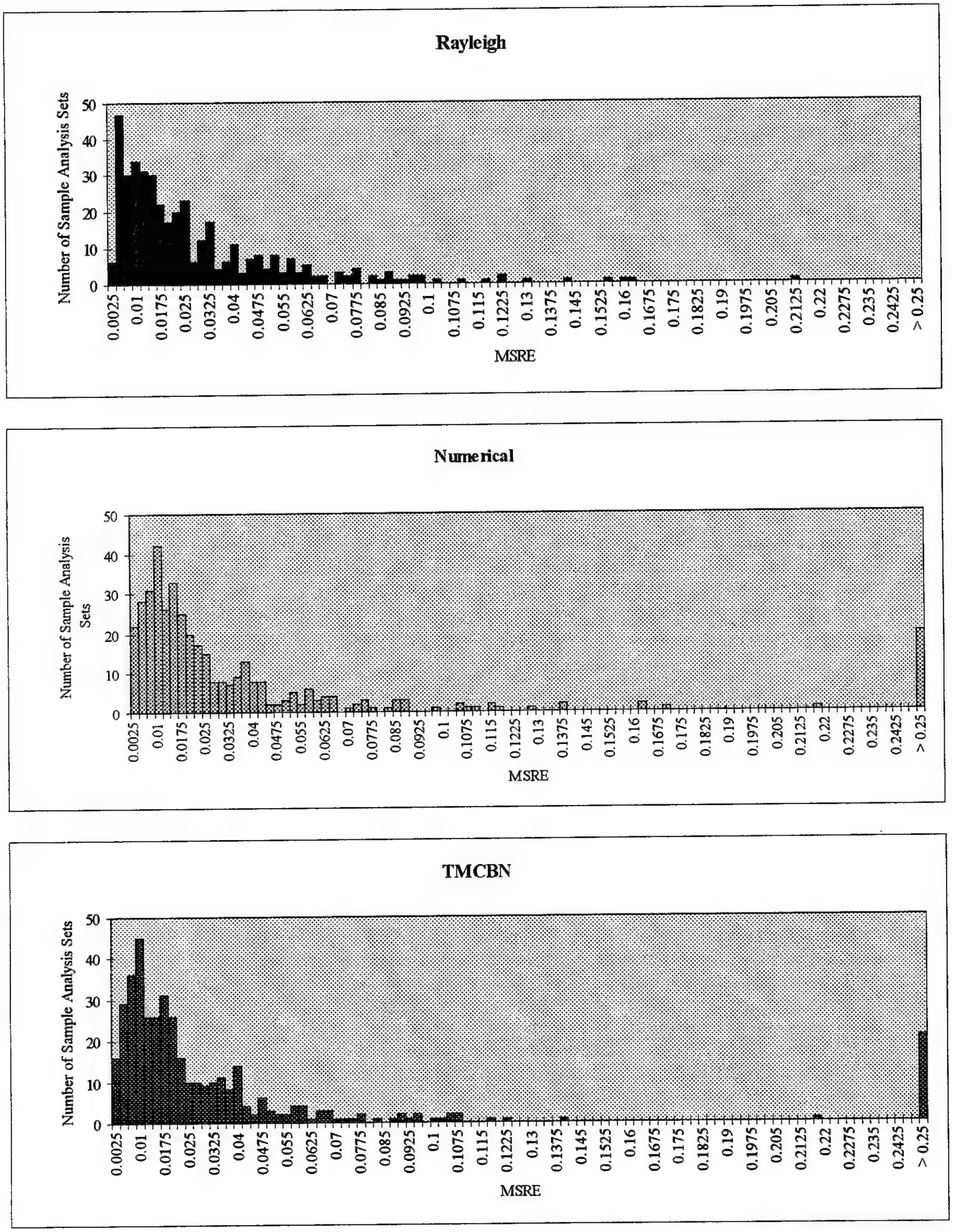


Figure 4.15 Overall Histograms Based on the Sample Analysis Set Results (continued)

These histograms basically mirrored the design point results. For *Numerical* and

TMCBN, however, the number of sample analysis sets with relatively high MSRE is proportionally larger than the number of design points with relatively high MSRE.

We next present statistics and CDF plots for our overall sample analysis set results:

Statistics	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
Average MSRE	0.0392	0.0284	0.6646	0.0325	0.0267	0.0273	0.9827	1.2813
Corresponding MRE Approximation	0.1981	0.1686	0.8152	0.1804	0.1633	0.1651	0.9913	1.132
AE (True CEP was 100)	±19.81	±16.86	±81.524	±18.04	±16.33	±16.509	±99.133	±113.2

Table 4.9 Overall Sample Analysis Set Statistics

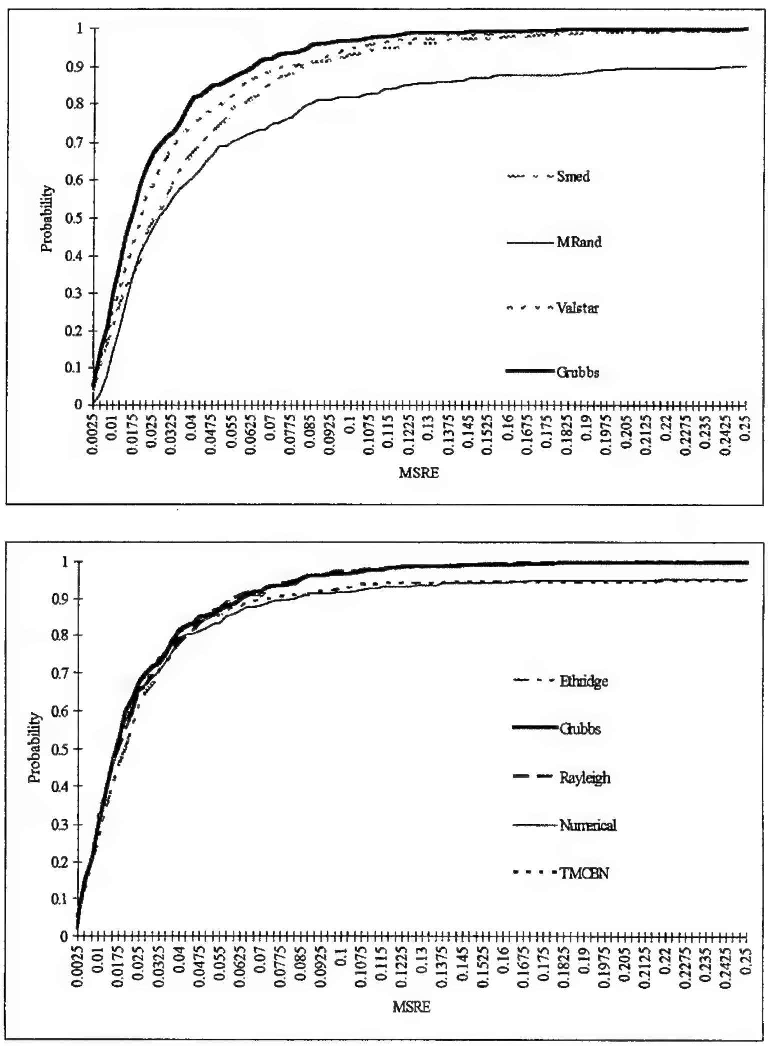


Figure 4.16 Overall CDF Plots Based on Our Sample Analysis Set Results

The upper graph in Figure 4.16 shows that *Grubbs* is stochastically dominated by *MRand, Smed,* and *Valstar*, while the lower graph displays the mixed results of *Grubbs* and the other four CEP estimators under consideration. Note in this lower graph that the two numerical estimators "separate" from the others around probability 0.9. Based on this CDF plot, the MSRE statistics, and the histograms presented, clearly there are some sample analysis set cases where *Numerical* and *TMCBN* performed more poorly than they had for the corresponding design point cases.

As we had done earlier for the design point results, we produced correlation tables results to determine which factors appeared most significant for our sample analysis sets:

CORRELATIONS (PEARSON)

	N	BIAS	P	SD_RATIO	SMED	ETHRIDGE	MRAND	VALSTAR	GRUBBS
BIAS	0.0000								
P	0.0000	0.0000							
SD_RATIO	0.0000	-0.0000	-0.0000						
SMED	-0.3864	-0.2744	0.2328	0.0700					
ETHRIDGE	-0.4596	-0.3260	0.2457	0.0694	0.9062				
MRAND	-0.1523	0.2089	0.0623	-0.0534	0.0210	0.0139			
VALSTAR	-0.4280	-0.1848	0.2799	0.1491	0.8996	0.8798	0.0322		
GRUBBS	-0.4639	-0.1929	0.2254	0.1354	0.9092	0.9082	0.0389	0.9828	
RAYLEIGH	-0.4173	-0.3094	0.2224	0.1238	0.9210	0.9419	0.0069	0.9292	0.9534

CASES INCLUDED 400 MISSING CASES 0

CORRELATIONS (PEARSON)

	N	BIAS	P	SD_RATIO	NUMERICAL
BIAS	0.0047				
P	-0.0187	-0.0072			
SD_RATIO	-0.0090	-0.0210	-0.0105		
NUMERICAL	-0.0838	0.1045	0.0264	-0.0571	
TMCBN	-0.0794	0.0995	0.0191	-0.0545	0.9947

CASES INCLUDED 384 MISSING CASES 16

Table 4.10 Statistix Correlation Tables for the Sample Analysis Set Results

Unlike the 1,100 design points, the 400 sample analysis sets are balanced with regard to the sample statistic ranges which define the sets. Therefore, in the upper table all correlation values are zero between these sample statistic range "factors." The 16 nonconvergent cases for *Numerical* and *TMCBN* account for the nonzero correlation values between the sample statistic ranges in the lower table.

Considering the results from these tables, sample size and sample bias appeared to be the most significant factors, just as sample size and bias were most significant for our design point results. There were differences, however. Sample correlation appeared more significant for our sample analysis set results than correlation had for our design point results, ranging from 0.0191(*TMCBN*) up to 0.2799 (*Valstar*). We also noted that for our sample analysis set results the two numerical estimators appeared relatively uncorrelated to any factor.

Based on these correlation table results, we decided to examine the sample analysis sets according to specific sample size and sample bias ranges in a manner similar to our analysis of the design point results. Due to their relatively poor performance in the overall CDF plots presented in Figure 4.16 and the statistics presented in Table 4.8, *MRand, Smed,* and *Valstar* were not considered in our casewise analysis of the sample analysis sets.

We begin our examination of the sample size case results by presenting for each estimator the average MSRE and the MRE and AE approximations that correspond to these MSRE averages. Because of their similar results, we present the sample size cases of fifteen, nine, and six as one group:

Sample Size = 15, 9, or 6	*Ethridge*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
Average MSRE	0.0213	0.0184	0.0208	0.0184	0.0172
Corresponding MRE Approximation	0.1459	0.1356	0.1444	0.1356	0.1311
AE (True CEP was 100)	±14.589	±13.558	±14.435	±13.564	±13.109

Sample Size = 3	*Ethridge*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
Average MSRE	0.0498	0.0515	0.0465	4.4268	5.7961
Corresponding MRE Approximation	0.2232	0.227	0.2156	2.104	2.4075
AE (True CEP was 100)	±22.315	±22.699	±21.565	±210.4	±240.75

Table 4.11 Statistics for Sample Analysis Set Sample Size Cases

From these statistics, the two numerical estimators show abysmal performance at sample size three. We next consider CDF plots for the sample analysis set data categorized by sample size:

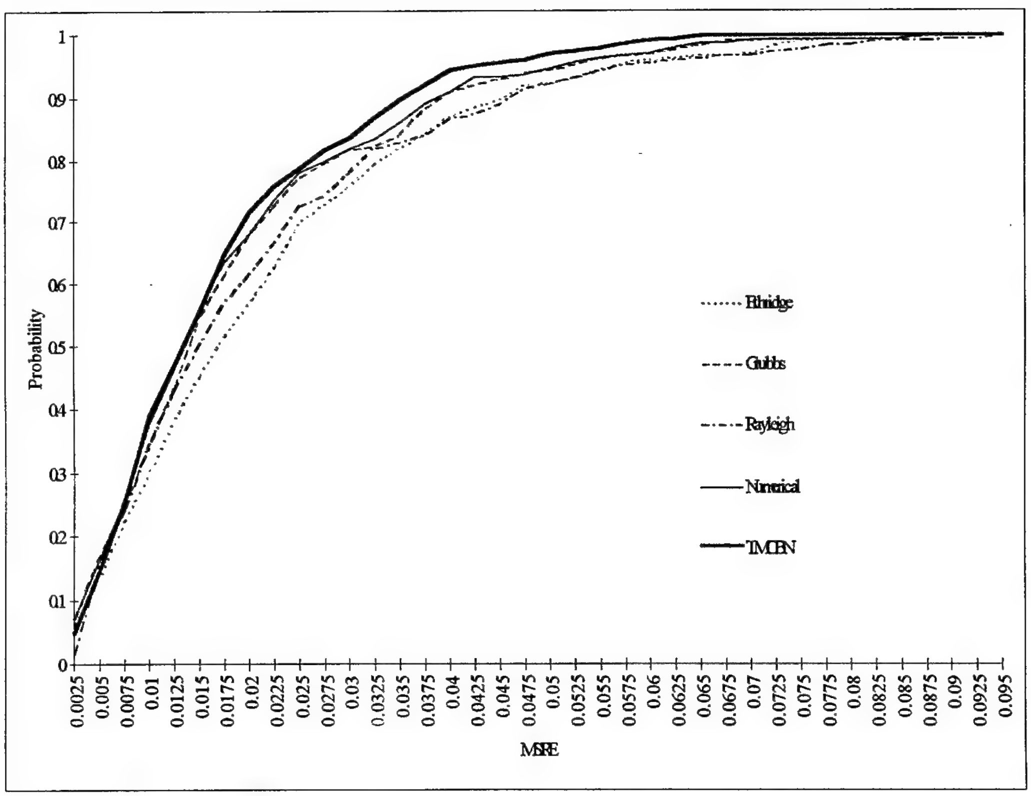
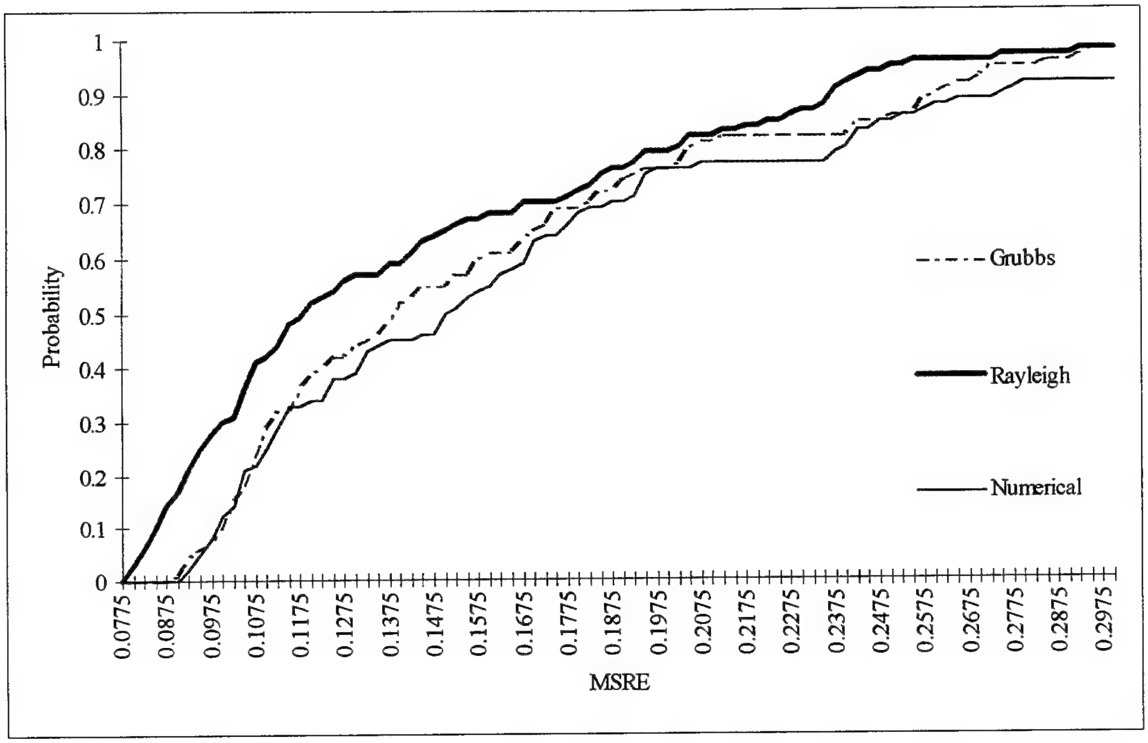


Figure 4.17 CDF Plot Based on Sample Analysis Set Results (n = 15, 9, or 6)



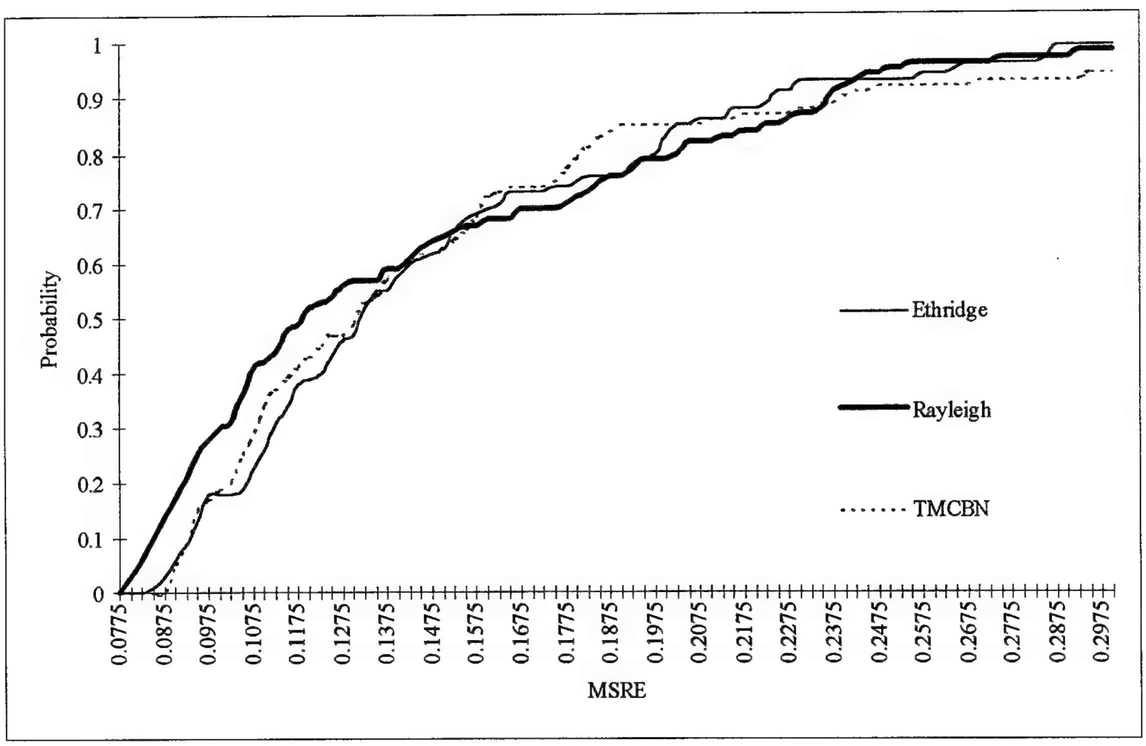


Figure 4.18 CDF Plots Based on Sample Analysis Set Results (n = 3)

Our CDF plots correspond to the statistics presented earlier for these two sample size sets. *TMCBN* performed marginally better at sample sizes fifteen, nine and six, while at sample size three *TMCBN, Rayleigh* and *Ethridge* performed best.

We next present statistics and CDF plots relating to sample bias range cases. Our CDF plots for the sample bias range cases under $2.75\bar{\sigma}$ provided no insights; for this reason we present these cases in a single plot.

Sample Bias = $[0, 0.75\bar{\sigma}]$	*Ethridge*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
Average MSRE	0.0383	0.0321	0.0384	0.0307	0.0269
Corresponding MRE Approximation	0.1958	0.1793	0.196	0.1753	0.1639
AE (True CEP was 100)	±19.577	±17.926	±19.598	±17.532	±16.387

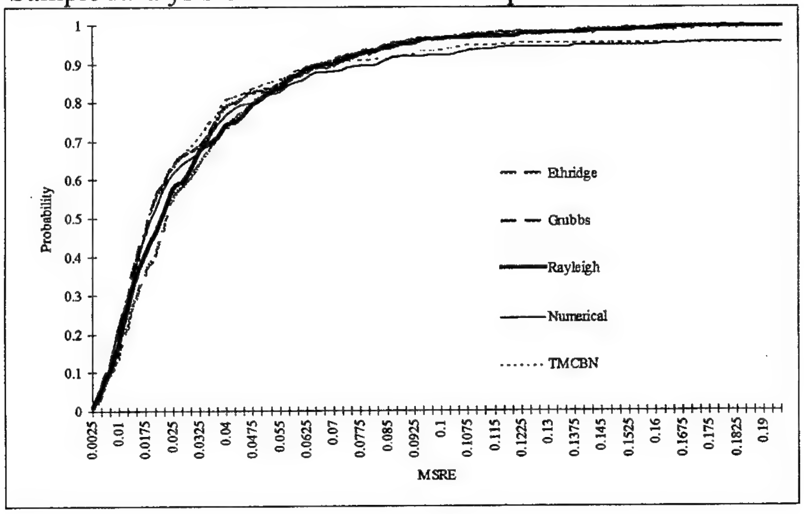
Sample Bias = $(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	*Ethridge*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
Average MSRE	0.035	0.0289	0.0309	0.0289	0.0268
Corresponding MRE Approximation	0.1872	0.1699	0.1758	0.1701	0.1637
AE (True CEP was 100)	±18.718	±16.994	±17.576	±17.014	±16.366

Sample Bias = $(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	*Ethridge*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
Average MSRE	0.0261	0.0298	0.0257	0.034	0.0317
Corresponding MRE Approximation	0.1615	0.1727	0.1602	0.1843	0.1781
AE (True CEP was 100)	±16.148	±17.271	±16.019	±18.428	±17.808

Sample Bias $> 2.75\bar{\sigma}$	*Ethridge*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
Average MSRE	0.0142	0.0158	0.0141	3.8774	5.0926
Corresponding MRE Approximation	0.1192	0.1258	0.1185	1.9691	2.2567
AE (True CEP was 100)	±11.92	±12.582	±11.855	±196.91	±225.67

Table 4.12 Statistics for Sample Analysis Set Sample Bias Cases

Sample Analysis Set Results When Sample Bias Was ≤ $2.75\bar{\sigma}$:



Sample Analysis Set Results When Sample Bias Was > $2.75\bar{\sigma}$:

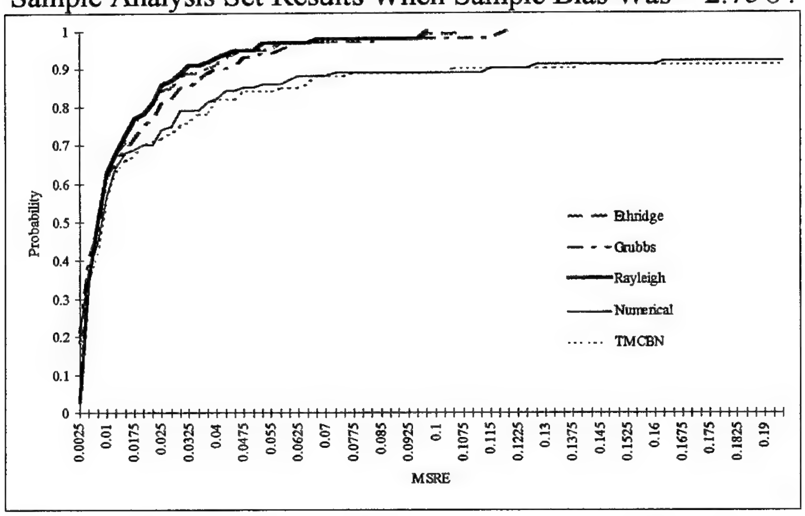


Figure 4.19 Sample Analysis Set CDF Plots Based on Sample Bias Range

There appears little disparity between the statistics and CDF plots of the competing estimators except for the case of sample bias greater than $2.75\bar{\sigma}$, where *Rayleigh* appears best and *Numerical and TMCBN* perform notably poorly

We next present the average MSRE and the AE results for each CEP estimator which correspond to each sample size/sample bias range combination. The corresponding

design point tables are also displayed again to allow comparison between the two results:

Average MSRE Results for Sample Analysis Set Sample Size/Sample Bias Combinations:

n	bîas	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
15	$[0,0.75\bar{\sigma}]$	0.0324	0.0193	0.0129	0.0161	0.0139	0.0211	0.0134	0.0131
15	$(0.75\bar{\sigma},1.25\bar{\sigma}]$	0.0224	0.0157	0.0157	0.0208	0.0131	0.0171	0.0119	0.0114
15	$(1.25\bar{\sigma},2.75\bar{\sigma}]$	0.0119	0.0114	0.0112	0.0139	0.009	0.0087	0.0081	0.0078
15	$(>2.75\bar{\sigma})$	0.0033	0.0029	0.0673	0.0028	0.0025	0.004	0.0023	0.0024
9	$[0,0.75\bar{\sigma}]$	0.0469	0.0319	0.0209	0.0255	0.0233	0.032	0.0226	0.0207
9	$(0.75\bar{\sigma},1.25\bar{\sigma}]$	0.0367	0.0258	0.0226	0.0287	0.0209	0.0232	0.0196	0.0188
9	$(1.25\bar{\sigma},2.75\bar{\sigma}]$	0.0283	0.0172	0.0248	0.0302	0.0202	0.0182	0.0191	0.0171
9	$(>2.75\bar{\sigma})$	0.0089	0.0066	0.1195	0.0081	0.0069	0.0072	0.0069	0.0069
6	$[0,0.75\bar{\sigma}]$	0.0566	0.0466	0.0321	0.0394	0.0365	0.046	0.0356	0.0318
6	$(0.75\bar{\sigma},1.25\bar{\sigma}]$	0.0451	0.0388	0.029	0.0355	0.0279	0.0322	0.0264	0.0244
6	$(1.25\bar{\sigma},2.75\bar{\sigma}]$	0.0356	0.0256	0.0369	0.0426	0.0305	0.0266	0.0293	0.0257
6	$(>2.75\bar{\sigma})$	0.0198	0.0136	0.516	0.0188	0.016	0.0138	0.0254	0.0261
3	$[0,0.75\bar{\sigma}]$	0.0786	0.0554	0.0442	0.0593	0.0548	0.0545	0.0552	0.0448
3	$(0.75\bar{\sigma},1.25\bar{\sigma}]$	0.0767	0.0598	0.0534	0.0637	0.0536	0.0511	0.0633	0.0574
3	$(1.25\bar{\sigma},2.75\bar{\sigma}]$	0.0793	0.0501	0.0636	0.0725	0.0597	0.0492	0.0854	0.0824
3	$(>2.75\bar{\sigma})$	0.0453	0.0338	9.5637	0.0428	0.038	0.0312	18.374	24.146

Average MSRE Results for Design Point Sample Size/Bias Combinations:

n	bias	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
15	0	0.0444	0.0259	0.0145	0.0177	0.017	0.0268	0.0167	0.0163
15	0.5σ	0.0382	0.023	0.0177	0.0229	0.0175	0.0266	0.0168	0.0161
15	1.0σ	0.0245	0.0178	0.0178	0.0228	0.0146	0.0176	0.0131	0.0127
15	2.0σ	0.0112	0.0115	0.0101	0.0123	0.0087	0.0079	0.008	0.008
15	4.0σ	0.0035	0.0033	0.0834	0.0027	0.0025	0.0045	0.0023	0.0023
9	0	0.0597	0.0455	0.0286	0.0352	0.0316	0.0462	0.0311	0.0268
9	0.5σ	0.0629	0.0379	0.0354	0.0445	0.0346	0.0429	0.0334	0.0301
9	1.0σ	0.0464	0.0308	0.0341	0.042	0.0295	0.0302	0.0277	0.0252
9	2.0σ	0.0272	0.019	0.0177	0.0223	0.0171	0.015	0.0166	0.0163
9	4.0σ	0.0065	0.0061	0.1556	0.0053	0.0052	0.0068	0.005	0.0056
6	0	0.1135	0.087	0.0664	0.0839	0.069	0.0881	0.068	0.0573
6	0.5σ	0.0929	0.0693	0.0631	0.0779	0.0645	0.0731	0.0627	0.0531
6	1.0σ	0.059	0.046	0.049	0.0591	0.0447	0.0436	0.0421	0.0385
6	2.0σ	0.0385	0.0299	0.0314	0.0366	0.0291	0.0256	0.0288	0.0281
6	4.0σ	0.0109	0.0096	0.7489	0.0094	0.0091	0.01	0.0091	0.0105
3	0	0.2186	0.1408	0.1458	0.1841	0.1584	0.1408	0.1564	0.1334
3	0.5σ	0.1945	0.1255	0.1517	0.1641	0.1402	0.128	0.152	0.1302
3	1.0σ	0.1326	0.0988	0.168	0.1239	0.1029	0.0926	0.1034	0.0927
3	2.0σ	0.0816	0.0576	0.9567	0.0716	0.06	0.0499	0.0623	0.0612
3	4.0σ	0.0241	0.0185	25.366	0.0182	0.0175	0.017	0.0712	0.1008

Table 4.13 Comparison of Design Point and Sample Analysis Set Average MSRE Results

Approximate Estimation Error (AE) Based on Sample Analysis Set Average MSRE Results for Each Sample Size/Sample Bias Case (Actual CEP was 100):

n	bias	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
15	[0,0.75$\bar{\sigma}$]	±18	±13.892	±11.358	±12.704	±11.792	±14.537	±11.593	±11.428
15	(0.75$\bar{\sigma}$,1.25$\bar{\sigma}$]	±14.967	±12.53	±12.53	±14.435	±11.458	±13.087	±10.931	±10.679
15	(1.25$\bar{\sigma}$,2.75$\bar{\sigma}$]	±10.909	±10.677	±10.583	±11.79	±9.4636	±9.3145	±9.0089	±8.8544
15	(>2.75$\bar{\sigma}$)	±5.7446	±5.3852	±25.942	±5.265	±4.9518	±6.2992	±4.7666	±4.9193
9	[0,0.75$\bar{\sigma}$]	±21.656	±17.861	±14.457	±15.969	±15.26	±17.891	±15.049	±14.376
9	(0.75$\bar{\sigma}$,1.25$\bar{\sigma}$]	±19.157	±16.062	±15.033	±16.941	±14.446	±15.217	±14.007	±13.724
9	(1.25$\bar{\sigma}$,2.75$\bar{\sigma}$]	±16.823	±13.115	±15.748	±17.378	±14.199	±13.477	±13.838	±13.069
9	(>2.75$\bar{\sigma}$)	±9.434	±8.124	±34.569	±9	±8.3211	±8.4829	±8.2873	±8.309
6	[0,0.75$\bar{\sigma}$]	±23.791	±21.587	±17.916	±19.849	±19.115	±21.457	±18.866	±17.819
6	(0.75$\bar{\sigma}$,1.25$\bar{\sigma}$]	±21.237	±19.698	±17.029	±18.841	±16.709	±17.933	±16.246	±15.632
6	(1.25$\bar{\sigma}$,2.75$\bar{\sigma}$]	±18.868	±16	±19.209	±20.64	±17.461	±16.308	±17.128	±16.019
6	(>2.75$\bar{\sigma}$)	±14.071	±11.662	±71.833	±13.711	±12.633	±11.761	±15.941	±16.168
3	[0,0.75$\bar{\sigma}$]	±28.036	±23.537	±21.024	±24.352	±23.409	±23.335	±23.491	±21.167
3	(0.75$\bar{\sigma}$,1.25$\bar{\sigma}$]	±27.695	±24.454	±23.108	±25.239	±23.152	±22.612	±25.165	±23.949
3	(1.25$\bar{\sigma}$,2.75$\bar{\sigma}$]	±28.16	±22.383	±25.219	±26.926	±24.434	±22.184	±29.225	±28.697
3	(>2.75$\bar{\sigma}$)	±21.284	±18.385	±309.25	±20.688	±19.494	±17.669	±428.65	±491.38

Approximate Estimation Error (AE) Based on Design Point Average MSRE Results for Each Sample Size/Bias Case (Actual CEP was 100):

n	bias	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
15	0	±21.071	±16.093	±12.042	±13.304	±13.038	±16.371	±12.923	±12.767
15	0.5σ	±19.545	±15.166	±13.304	±15.133	±13.229	±16.31	±12.961	±12.689
15	1.0σ	±15.652	±13.342	±13.342	±15.1	±12.083	±13.266	±11.446	±11.269
15	2.0σ	±10.583	±10.724	±10.05	±11.091	±9.3274	±8.8882	±8.9443	±8.9443
15	4.0σ	±5.9161	±5.7446	±28.879	±5.1962	±5	±6.7082	±4.7958	±4.7958
9	0	±24.434	±21.331	±16.912	±18.762	±17.776	±21.494	±17.635	±16.371
9	0.5σ	±25.08	±19.468	±18.815	±21.095	±18.601	±20.712	±18.276	±17.349
9	1.0σ	±21.541	±17.55	±18.466	±20.494	±17.176	±17.378	±16.643	±15.875
9	2.0σ	±16.492	±13.784	±13.304	±14.933	±13.077	±12.247	±12.884	±12.767
9	4.0σ	±8.0623	±7.8102	±39.446	±7.2801	±7.2111	±8.2462	±7.0711	±7.4833
6	0	±33.69	±29.496	±25.768	±28.965	±26.268	±29.682	±26.077	±23.937
6	0.5σ	±30.48	±26.325	±25.12	±27.911	±25.397	±27.037	±25.04	±23.043
6	1.0σ	±24.29	±21.448	±22.136	±24.31	±21.142	±20.881	±20.518	±19.621
6	2.0σ	±19.621	±17.292	±17.72	±19.131	±17.059	±16	±16.971	±16.763
6	4.0σ	±10.44	±9.798	±86.539	±9.6954	±9.5394	±10	±9.5394	±10.247
3	0	±46.755	±37.523	±38.184	±42.907	±39.799	±37.523	±39.547	±36.524
3	0.5σ	±44.102	±35.426	±38.949	±40.509	±37.443	±35.777	±38.987	±36.083
3	1.0σ	±36.414	±31.432	±40.988	±35.199	±32.078	±30.43	±32.156	±30.447
3	2.0σ	±28.566	±24	±97.811	±26.758	±24.495	±22.338	±24.96	±24.739
3	4.0σ	±15.524	±13.601	±503.65	±13.491	±13.229	±13.038	±26.683	±31.749

Table 4.14 Comparison of Design Point and Sample Analysis Set AE Results

We conclude our presentation of our sample analysis set results by displaying RE plots which correspond to those presented for the design point results in Figures 4.8 through 4.11:

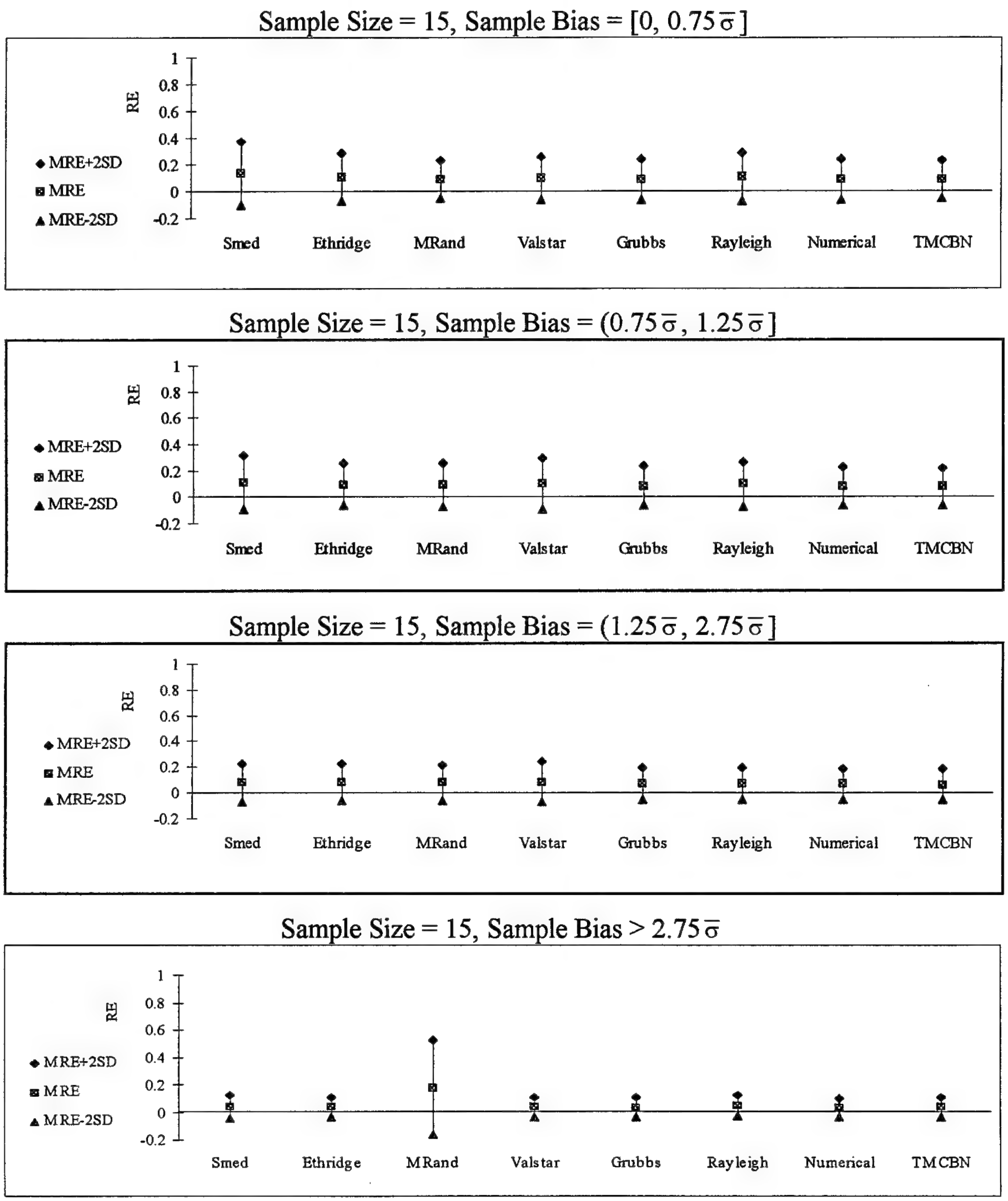
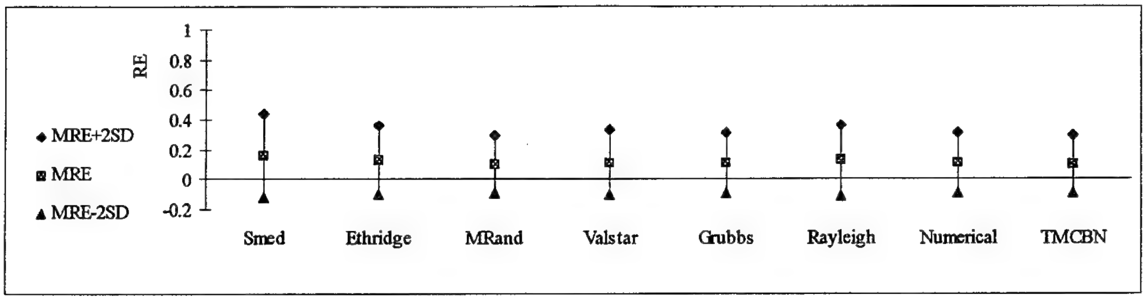
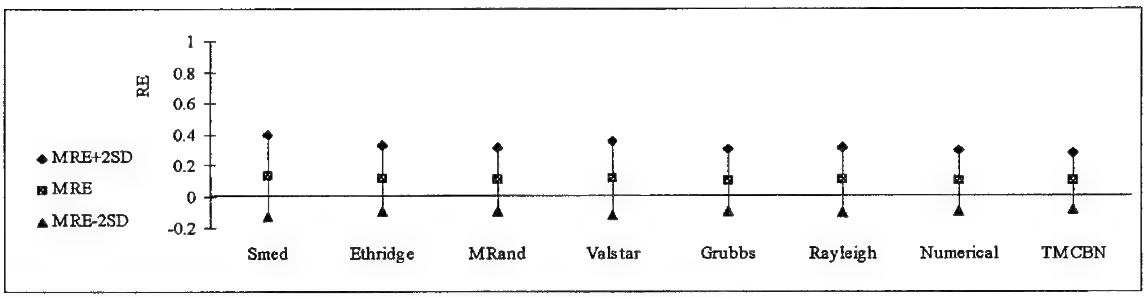


Figure 4.20 Sample Analysis Set RE Plots for Each bîas Case for Sample Size Fifteen

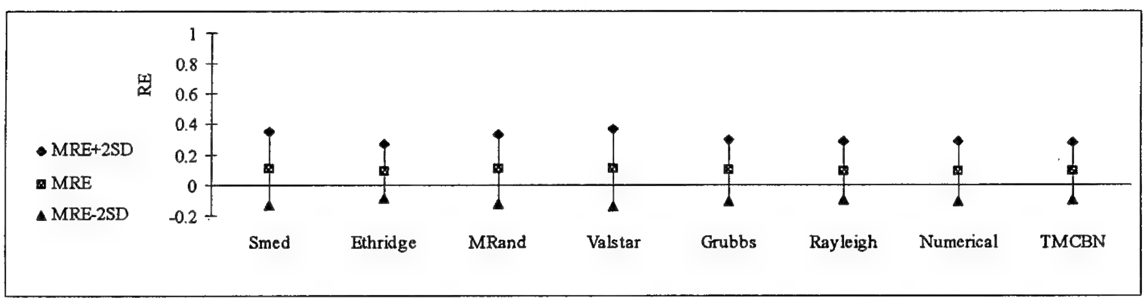
Sample Size = 9, Sample Bias = [0, 0.75$\bar{\sigma}$]



Sample Size = 9, Sample Bias = (0.75$\bar{\sigma}$, 1.25$\bar{\sigma}$]



Sample Size = 9, Sample Bias = (1.25$\bar{\sigma}$, 2.75$\bar{\sigma}$]



Sample Size = 9, Sample Bias > 2.75$\bar{\sigma}$

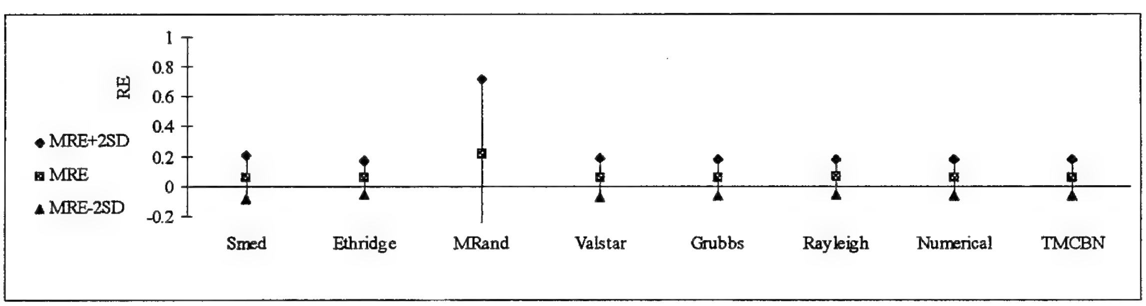


Figure 4.21 Sample Analysis Set RE Plots for Each bîas Case for Sample Size Nine

Sample Size = 6, Sample Bias = [0, 0.75$\bar{\sigma}$]

Sample Size = 6, Sample Bias = (0.75$\bar{\sigma}$, 1.25$\bar{\sigma}$]

Sample Size = 6, Sample Bias = (1.25$\bar{\sigma}$, 2.75$\bar{\sigma}$]

Sample Size = 6, Sample Bias > 2.75$\bar{\sigma}$

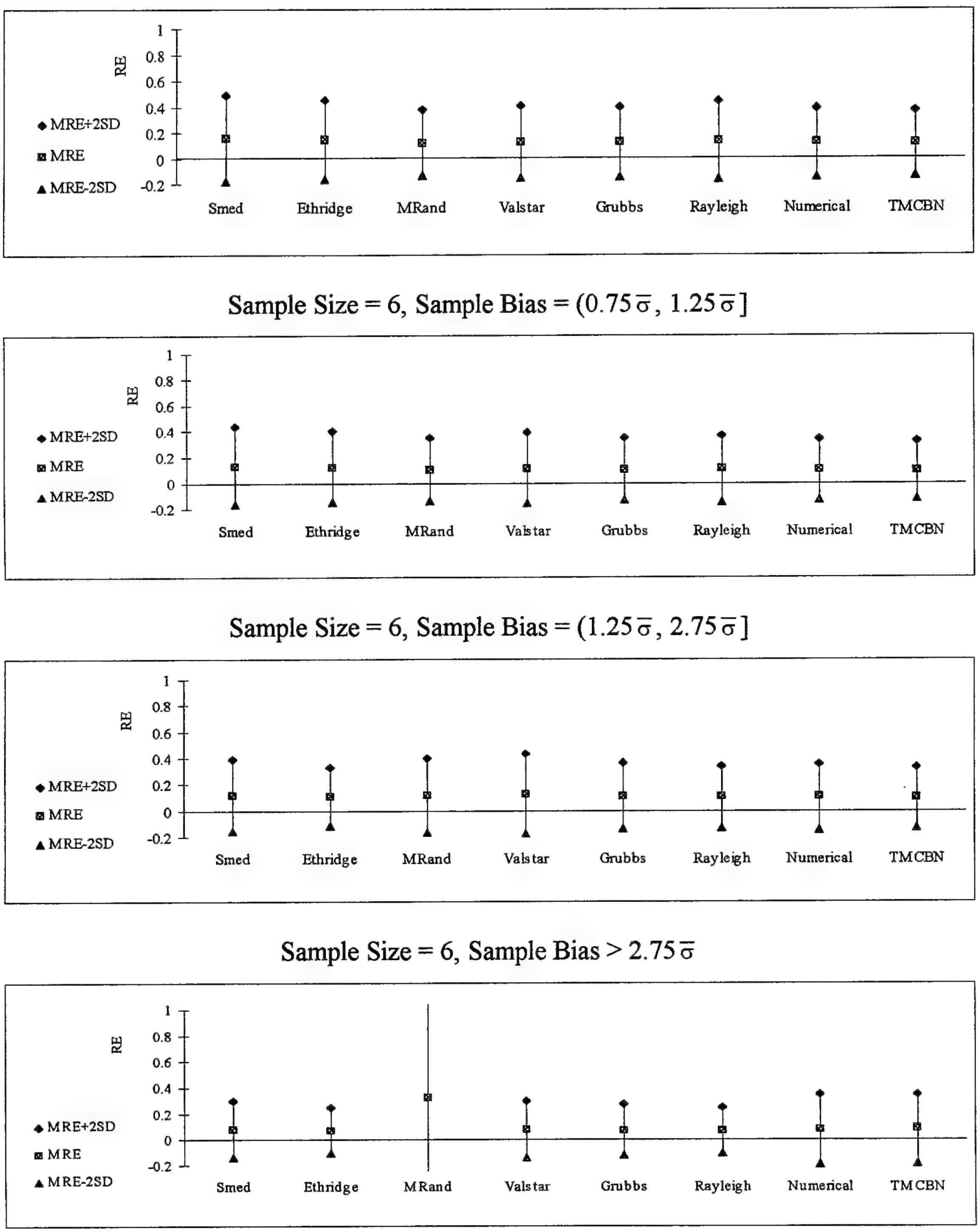
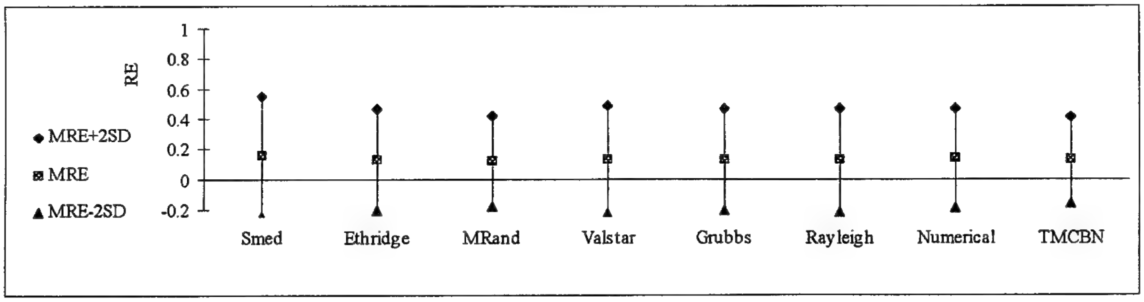
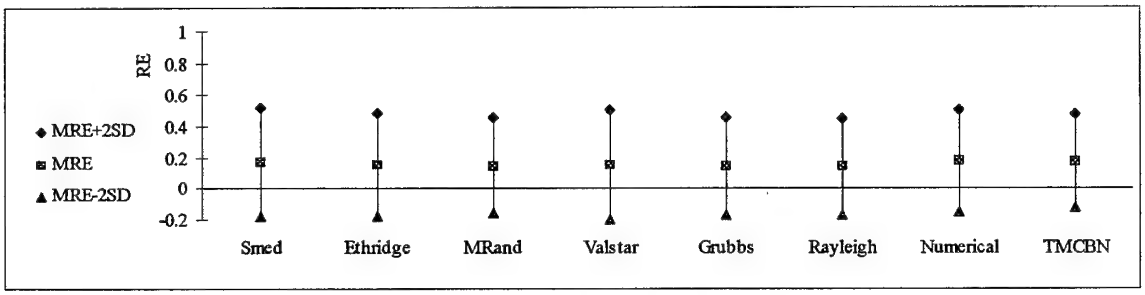


Figure 4.22 Sample Analysis Set RE Plots for Each bîas Case for Sample Size Six

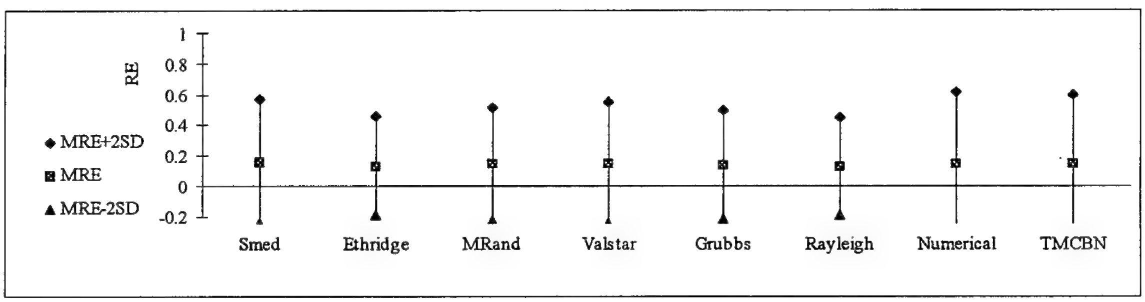
Sample Size = 3, Sample Bias = [0, 0.75 $\overline{\sigma}$]



Sample Size = 3, Sample Bias = (0.75 $\overline{\sigma}$, 1.25 $\overline{\sigma}$]



Sample Size = 3, Sample Bias = (1.25 $\overline{\sigma}$, 2.75 $\overline{\sigma}$]



Sample Size = 3, Sample Bias > 2.75 $\overline{\sigma}$

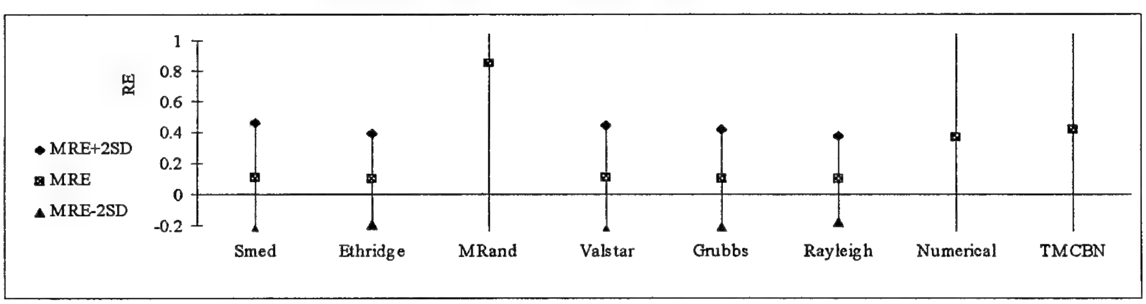


Figure 4.23 Sample Analysis Set RE Plots for Each bîas Case for Sample Size Three

These plots show that our sample analysis set results closely resembled those of our design points. Again, the difference in MRE and VRE scores for the eight CEP estimators considered was marginal for most cases. The one noticeable difference in our sample analysis set results, however, is the increased MRE and VRE of the two numerical estimators for sample size three and sample bias greater than $2.75\overline{\sigma}$.

This concludes our presentation of the results of our simulation experiment. The overall and casewise statistics, CDF plots, and MRE plots shown in this chapter provide the basis for our recommendations presented in Chapter 5.

V. CONCLUSIONS AND RECOMMENDATIONS

In this final chapter, we summarize our findings and make recommendations based on the results of the simulation experiment. We first highlight the more notable observations made during the experiment. We then address how our results relate to the objectives listed in Section 1.4 and finally conclude by presenting suggestions for further study.

1. Nonparametric CEP Estimators: Smith (1982) claimed that nonparametric CEP estimators were not suitable for small samples. We tested this claim by considering the nonparametric sample median estimator. Based on the weak performance of *Smed*, we concur with Smith.

2. Numerical Integration Methods: Elder [1986, page 5-4] and Puhek [1992, page 2-7] assumed (but did not test) that a numerical integration CEP estimator, while requiring excessive computation time, was generally always superior to a non-numerical CEP estimator. While *Numerical* and *TMCBN* do require more computation time than CEP estimators from other categories, it is hardly prohibitive on today's Pentium equipped computers (a sample of 15 coordinates is usually computed in under a minute). Our results and the earlier results of Tongue (1993) show, however, that there are numerous instances where the non-numerical estimators outperformed *Numerical* or *TMCBN*.

Tongue claimed that *CBN (Numerical)* and *TMCBN* dominated for sample sizes greater than ten. Our sample analysis set MSRE results for sample size fifteen supports this assertion.

3. MOEs: With regard to the MOEs used in our experiment, we noticed in our results

that the CEP estimator which was best for MSRE was also usually best for MRE and/or VRE. Since MSRE combines aspects of both MRE and VRE, we recommend MSRE if only one MOE is used in a comparison of CEP estimators.

4. Estimators Not Considered in Previous Studies: Two of the CEP estimators compared in this study, *Smed* and *Valstar*, were not considered in the previous works of Elder (1986), Puhek (1992), or Tongue (1993). Based on both our design point and sample analysis set results, neither of these estimators was highly competitive.

5.1 Results Based on the Thesis Objectives

In this section, we revisit the three thesis objectives first presented in Section 1.4.

Objective 1: The first of our objectives was to develop a tool which CEP analysts could use to determine which CEP estimator to use for small samples (15 or less) for a given set of conditions. We base our recommendations on the precision that required for the CEP estimate.

If maximum precision is required, we recommend the following decision grid based on the best average MSRE value for each sample analysis set sample size/sample bias combination:

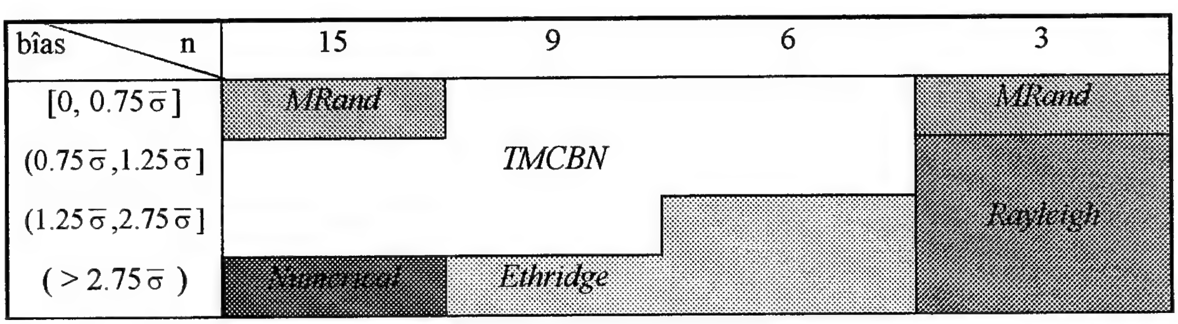


Figure 5.1 Recommended Decision Grid for Maximum Precision

We base our recommendations on our sample analysis set results rather than the design point results because the sample analysis sets model the real life scenario of estimating CEP based on only sample statistics.

If the decision maker is willing to tolerate relatively small increase in the amount of anticipated error, we can recommend more flexible choices. Recall that for a given case the AE for a given estimator refers to the approximate error one can expect from that estimator based on it's MSRE. Let MAE represent the minimum AE based on the sample analysis set values from Table 4.8 for each sample size/sample bias case. In the next three grids, we present CEP estimators which have AE values within 0.01*CEP, 0.02*CEP, and 0.05*CEP of the MAE respectively:

$\hat{\text{bias}}$ \ n	15	9	6	3
$[0, 0.75\bar{\sigma}]$	*MRand* *Grubbs* *Numerical* *TMCBN*	*MRand* *Grubbs* *Numerical* *TMCBN*	*MRand* *Numerical* *TMCBN*	*MRand* *TMCBN*
$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	*Grubbs* *Numerical* *TMCBN*	*Grubbs* *Numerical* *TMCBN*	*Numerical* *TMCBN*	*MRand* *Grubbs* *Rayleigh*
$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	*Grubbs* *Rayleigh* *Numerical* *TMCBN*	*Ethridge* *Rayleigh* *Numerical* *TMCBN*	*Rayleigh* *TMCBN*	*Ethridge* *Rayleigh*
$(> 2.75\bar{\sigma})$	*Smed* *Ethridge* *Valstar* *Grubbs* *Numerical* *TMCBN*	*Ethridge* *Valstar* *Grubbs* *Rayleigh* *Numerical* *TMCBN*	*Ethridge* *Grubbs* *Rayleigh*	*Ethridge* *Rayleigh*

Figure 5.2 CEP Estimators Which Have AE Values Within 0.01*CEP of the MAE.

bîas \ n	15	9	6	3
[0, 0.75$\bar{\sigma}$]	*MRand* *Valstar* *Grubbs* *Numerical* *TMCBN*	*MRand* *Valstar* *Grubbs* *Numerical* *TMCBN*	*MRand* *Grubbs* *Numerical* *TMCBN*	*MRand* *TMCBN*
(0.75$\bar{\sigma}$,1.25$\bar{\sigma}$]	*Ethridge* *MRand* *Grubbs* *Numerical* *TMCBN*	*MRand* *Grubbs* *Rayleigh* *Numerical* *TMCBN*	*MRand* *Grubbs* *Numerical* *TMCBN*	*Ethridge* *MRand* *Grubbs* *Rayleigh* *TMCBN*
(1.25$\bar{\sigma}$,2.75$\bar{\sigma}$]	*Ethridge* *MRand* *Grubbs* *Rayleigh* *Numerical* *TMCBN*	*Ethridge* *Grubbs* *Rayleigh* *Numerical* *TMCBN*	*Grubbs* *Rayleigh* *Numerical* *TMCBN*	*Ethridge* *Rayleigh*
(>2.75$\bar{\sigma}$)	*Smed* *Ethridge* *Valstar* *Grubbs* *Rayleigh* *Numerical* *TMCBN*	*Smed* *Ethridge* *Valstar* *Grubbs* *Rayleigh* *Numerical* *TMCBN*	*Ethridge* *Grubbs* *Rayleigh*	*Ethridge* *Grubbs* *Rayleigh*

Figure 5.3 CEP Estimators Which Have AE Values Within 0.02*CEP of the MAE.

bîas \ n	15	9	6	3
[0, 0.75$\bar{\sigma}$]	*Smed*	*Smed*	*Smed*	*Smed*
(0.75$\bar{\sigma}$,1.25$\bar{\sigma}$]	-	*Smed*	-	*Smed*
(1.25$\bar{\sigma}$,2.75$\bar{\sigma}$]	-	-	-	*Smed* *Numerical* *TMCBN*
(>2.75$\bar{\sigma}$)	*MRand*	*MRand*	*MRand*	*MRand* *Numerical* *TMCBN*

Figure 5.4 CEP Estimators Which Do *Not* Have AE Values Within 0.05*CEP of the MAE.

We next recommend decision grids for the event in which the CEP analyst does not have access to a personal computer. The *Numerical* and *TMCBN* estimators both require a either a computer program manually written in a language such as FORTRAN or special software such as MathCAD to perform numerical approximation. In a situation where the CEP analyst does not have access to a personal computer and must estimate the CEP "by hand", we recommend the following decision grids based on our sample analysis set MSRE and AE results:

<table>
<tr><th>bîas \ n</th><th>15</th><th>9</th><th>6</th><th>3</th></tr>
<tr><td>[0, 0.75$\bar{\sigma}$]</td><td colspan="4">MRand</td></tr>
<tr><td>(0.75$\bar{\sigma}$,1.25$\bar{\sigma}$]</td><td colspan="3">Grubbs</td><td rowspan="3">Rayleigh</td></tr>
<tr><td>(1.25$\bar{\sigma}$,2.75$\bar{\sigma}$]</td><td>Rayleigh</td><td colspan="2" rowspan="2">Ethridge</td></tr>
<tr><td>(> 2.75$\bar{\sigma}$)</td><td>Grubbs</td></tr>
</table>

Figure 5.5 Recommended Decision Grid for Maximum Precision, Numerical Methods Excluded

bîas \ n	15	9	6	3
[0, 0.75$\bar{\sigma}$]	*MRand* *Grubbs*	*MRand* *Grubbs*	*MRand*	*MRand*
(0.75$\bar{\sigma}$,1.25$\bar{\sigma}$]	*Grubbs*	*MRand* *Grubbs* *Rayleigh*	*MRand* *Grubbs* *Rayleigh*	*MRand* *Grubbs* *Rayleigh*
(1.25$\bar{\sigma}$,2.75$\bar{\sigma}$]	*Grubbs* *Rayleigh*	*Ethridge* *Rayleigh*	*Ethridge* *Rayleigh*	*Ethridge* *Rayleigh*
(> 2.75$\bar{\sigma}$)	*Smed* *Ethridge* *Valstar* *Grubbs*	*Ethridge* *Valstar* *Grubbs* *Rayleigh*	*Ethridge* *Grubbs* *Rayleigh*	*Ethridge* *Rayleigh*

Figure 5.6 CEP Estimators Which Have AE Values Within 0.01*CEP of the MAE, Numerical Methods Excluded

bias \ n	15	9	6	3
[0, 0.75 $\bar{\sigma}$]	*MRand* *Valstar* *Grubbs*	*MRand* *Valstar* *Grubbs*	*MRand* *Valstar* *Grubbs*	*MRand*
(0.75 $\bar{\sigma}$,1.25 $\bar{\sigma}$]	*Ethridge* *MRand* *Grubbs* *Rayleigh*	*Ethridge* *MRand* *Grubbs* *Rayleigh*	*MRand* *Valstar* *Grubbs* *Rayleigh*	*Ethridge* *MRand* *Grubbs* *Rayleigh*
(1.25 $\bar{\sigma}$,2.75 $\bar{\sigma}$]	*Smed* *Ethridge* *MRand* *Grubbs* *Rayleigh*	*Ethridge* *Grubbs* *Rayleigh*	*Ethridge* *Grubbs* *Rayleigh*	*Ethridge* *Rayleigh*
(> 2.75 $\bar{\sigma}$)	*Smed* *Ethridge* *Valstar* *Grubbs* *Rayleigh*	*Smed* *Ethridge* *Valstar* *Grubbs* *Rayleigh*	*Ethridge* *Grubbs* *Rayleigh*	*Ethridge* *Grubbs* *Rayleigh*

Figure 5.7 CEP Estimators Which Have AE Values Within 0.02*CEP of the MAE, Numerical Estimators Excluded

bias \ n	15	9	6	3
[0, 0.75 $\bar{\sigma}$]	*Smed*	*Smed*	*Smed*	*Smed*
(0.75 $\bar{\sigma}$,1.25 $\bar{\sigma}$]	-	-	-	*Smed*
(1.25 $\bar{\sigma}$,2.75 $\bar{\sigma}$]	-	-	-	*Smed*
(> 2.75 $\bar{\sigma}$)	*MRand*	*MRand*	*MRand*	*MRand*

Figure 5.8 CEP Estimators Which Do *Not* Have AE Values Within 0.05*CEP of the MAE, Numerical Estimators Excluded

Finally, in addition to not having access to a personal computer, an analyst could face a situation where they not have access to the chi-square distribution tables required

for the *Grubbs* computation. In this case, the analyst only has the capability to perform the necessary calculations for *Smed, Ethridge, MRand, Valstar,* and *Rayleigh*. We recommend the following decision grids based on our sample analysis set MSRE and AE results for this instance:

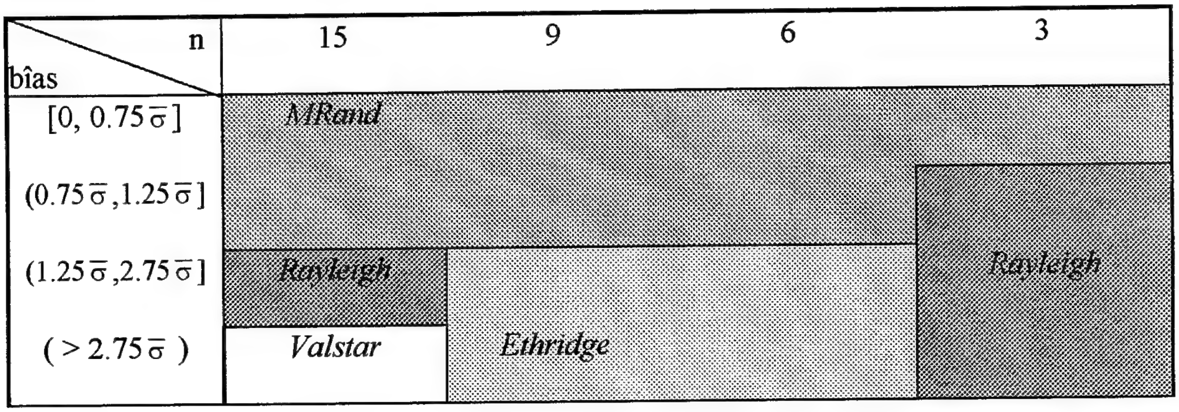


Figure 5.9 Recommended Decision Grid for Maximum Precision, *Grubbs, Numerical,* and *TMCBN* Excluded

bîas \ n	15	9	6	3
[0, 0.75$\bar{\sigma}$]	*MRand*	*MRand*	*MRand*	*MRand*
(0.75$\bar{\sigma}$,1.25$\bar{\sigma}$]	*MRand* *Ethridge* *Rayleigh*	*MRand* *Rayleigh*	*MRand* *Rayleigh*	*MRand* *Rayleigh*
(1.25$\bar{\sigma}$,2.75$\bar{\sigma}$]	*Rayleigh*	*Ethridge* *Rayleigh*	*Ethridge* *Rayleigh*	*Ethridge* *Rayleigh*
(> 2.75$\bar{\sigma}$)	*Smed* *Ethridge* *Valstar*	*Ethridge* *Valstar* *Rayleigh*	*Ethridge* *Rayleigh*	*Ethridge* *Rayleigh*

Figure 5.10 CEP Estimators Which Have AE Values Within 0.01*CEP of the MAE, *Grubbs, Numerical,* and *TMCBN* Excluded

n / bîas	15	9	6	3
[0, 0.75 $\bar{\sigma}$]	*MRand* *Valstar*	*MRand* *Valstar*	*MRand* *Valstar*	*MRand*
(0.75 $\bar{\sigma}$,1.25 $\bar{\sigma}$]	*Ethridge* *MRand* *Valstar* *Rayleigh*	*Ethridge* *MRand* *Valstar* *Rayleigh*	*MRand* *Valstar* *Rayleigh*	*Ethridge* *MRand* *Rayleigh*
(1.25 $\bar{\sigma}$,2.75 $\bar{\sigma}$]	*Smed* *Ethridge* *MRand* *Rayleigh*	*Ethridge* *Rayleigh*	*Ethridge* *Rayleigh*	*Ethridge* *Rayleigh*
(> 2.75 $\bar{\sigma}$)	*Smed* *Ethridge* *Valstar* *Rayleigh*	*Smed* *Ethridge* *Valstar* *Rayleigh*	*Ethridge* *Rayleigh*	*Ethridge* *Rayleigh*

Figure 5.11 CEP Estimators Which Have AE Values Within 0.02*CEP of the MAE, *Grubbs, Numerical,* and *TMCBN* Excluded

n / bîas	15	9	6	3
[0, 0.75 $\bar{\sigma}$]	*Smed*	*Smed*	*Smed*	*Smed*
(0.75 $\bar{\sigma}$,1.25 $\bar{\sigma}$]	-	-	-	*Smed*
(1.25 $\bar{\sigma}$,2.75 $\bar{\sigma}$]	-	-	-	*Smed*
(> 2.75 $\bar{\sigma}$)	*MRand*	*MRand*	*MRand*	*MRand*

Figure 5.12 CEP Estimators Which Do *Not* Have AE Values Within 0.05*CEP of the MAE, *Grubbs, Numerical,* and *TMCBN* Excluded

Objective 2: Our second objective was to compare similarities/differences in our sample analysis set and our design point results.

To accomplish this objective, we display grids for both the design point and sample analysis set results which list the estimators which can be expected to deliver an estimate

of within 0.01*CEP of the MEA:

CEP Estimators With an AE of Within 0.01*CEP of the Minimum AE,
Based on Our Sample Analysis Set Results

bîas \ n	15	9	6	3
[0, 0.75 $\bar{\sigma}$]	*MRand* *Grubbs* *Numerical* *TMCBN*	*MRand* *Grubbs* *Numerical* *TMCBN*	*MRand* *Numerical* *TMCBN*	*MRand* *TMCBN*
(0.75 $\bar{\sigma}$, 1.25 $\bar{\sigma}$]	*Grubbs* *Numerical* *TMCBN*	*Grubbs* *Numerical* *TMCBN*	*Numerical* *TMCBN*	*MRand* *Grubbs* *Rayleigh*
(1.25 $\bar{\sigma}$, 2.75 $\bar{\sigma}$]	*Grubbs* *Rayleigh* *Numerical* *TMCBN*	*Ethridge* *Rayleigh* *Numerical* *TMCBN*	*Rayleigh* *TMCBN*	*Ethridge* *Rayleigh*
(> 2.75 $\bar{\sigma}$)	*Smed* *Ethridge* *Valstar* *Grubbs* *Numerical* *TMCBN*	*Ethridge* *Valstar* *Grubbs* *Rayleigh* *Numerical* *TMCBN*	*Ethridge* *Grubbs* *Rayleigh*	*Ethridge* *Rayleigh*

CEP Estimators With an AE of Within 0.01*CEP of the Minimum AE,
Based on Our Design Point Results

bîas \ n	15	9	6	3
0σ	*MRand* *Numerical* *TMCBN*	*MRand* *TMCBN*	*TMCBN*	*Ethridge* *Rayleigh* *TMCBN*
0.5σ	*MRand* *Grubbs* *Numerical* *TMCBN*	*Numerical* *TMCBN*	*TMCBN*	*Ethridge* *Rayleigh* *TMCBN*
1.0σ	*Grubbs* *Numerical* *TMCBN*	*Numerical* *TMCBN*	*Numerical* *TMCBN*	*Ethridge* *Rayleigh* *TMCBN*
2.0σ	*Grubbs* *Rayleigh* *Numerical* *TMCBN*	*Grubbs* *Rayleigh* *Numerical* *TMCBN*	*Rayleigh* *Numerical* *TMCBN*	*Rayleigh*
4.0σ	*Ethridge* *Valstar* *Grubbs* *Numerical* *TMCBN*	*Smed* *Ethridge* *Valstar* *Grubbs* *Numerical* *TMCBN*	*Smed* *Ethridge* *Valstar* *Grubbs* *Rayleigh* *Numerical*	*Ethridge* *Valstar* *Grubbs* *Rayleigh*

Figure 5.13 Grids Summarizing Our Sample Analysis Set and Design Point Results

Rayleigh, Grubbs, Numerical, and *TMCBN* appear the most robust in the sense that they all appear in the majority of the grid squares in Figure 5.13. The CEP estimators which are in both grids for similar corresponding cases are shaded; in a case by case comparison between the two grids, it is clear that the results are almost identical.

Objective 3: Our third objective was to compare our results with those of earlier studies. As alluded to in Chapter 4, our results were similar to those of Elder (1985) and Tongue (1993), while different than those of Puhek (1992). As mentioned in Section 4.2, we believe that the primary reason for this difference is the fact that our design point estimates were an average of ten runs, while Puhek did not conduct replications at his design points. Using single replications would mean greater variance and less accuracy in Puhek's results. The fact that our results did closely resemble those of the independent studies of Elder and Tongue contributes to the verification of our results.

5.2 Recommendations for Further Study

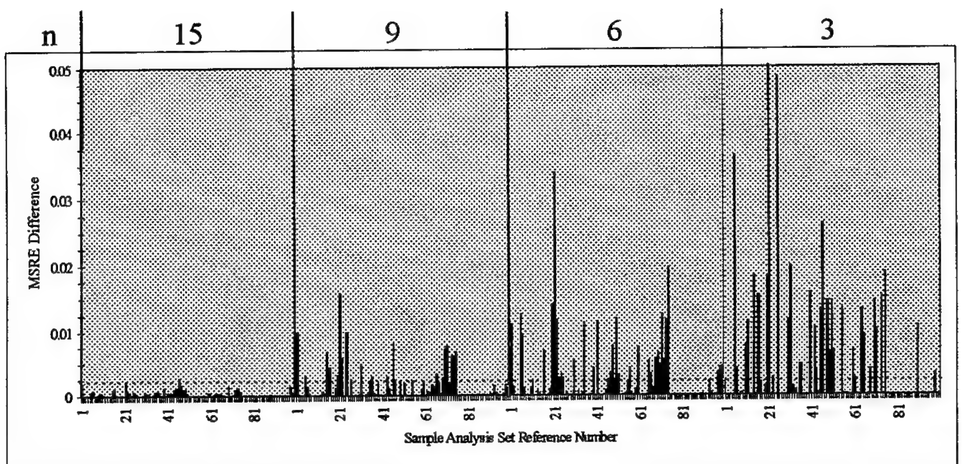
We conclude our paper with specific recommendations based on our results for further areas of study. Tongue asserted that the future expense of new ballistic weapon systems will make extensive testing prohibitive, resulting in accuracy analysts trying to accurately predict CEP with smaller and smaller sets of data [1993, page 7-8]. Tongue believed that due to this expense, further research was required in two areas:

1. Developing new CEP estimators with greater accuracy for small samples.
2. Establishing a more detailed decision grid for use in selecting the proper CEP estimator for a given set of circumstances.

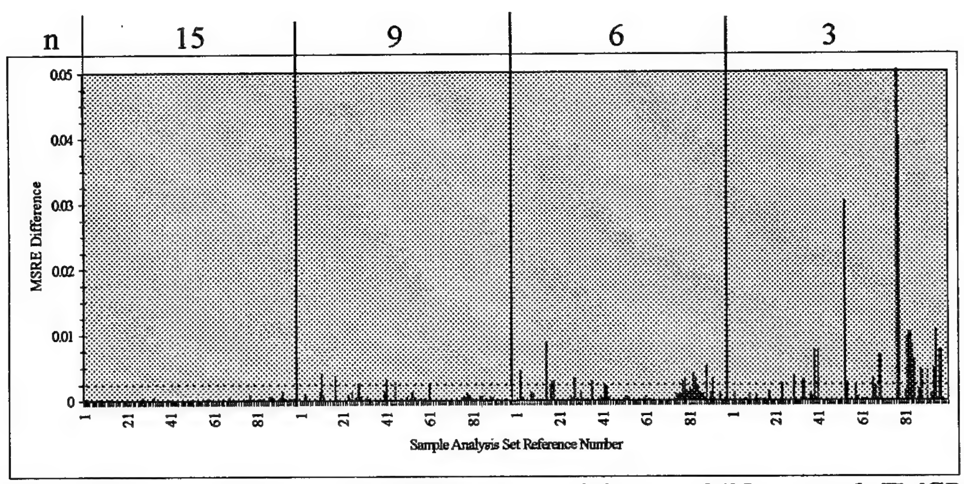
Our recommendations for additional study focus on these two points:

1. Developing Modified Estimators: Tongue [1993, page 7-6] recommended that improved modifications of *Rayleigh, Grubbs,* and *MRand* be developed. Tongue based this recommendation on the strong performance of *TMCBN,* his modification of the *CBN* (*Numerical*) CEP estimator, in his thesis simulation experiment.

We also independently verified that *TMCBN* generally outperformed the *CBN* (*Numerical)* estimator:



MSRE Difference Between *Numerical* and the Minimum of (*Numerical*, *TMCBN*)



MSRE Difference Between *TMCBN* and the Minimum of (*Numerical*, *TMCBN*

Figure 5.14 Magnitude of the MSRE Difference Between *Numerical*, *TMCBN* ,and the Minimum MSRE of (*Numerical*, *TMCBN*)

The bar graphs in Figure 5.14 indicate that the estimation difference between *Numerical* and *TMCBN* tends to increase as sample size decreases, with *TMCBN* generally achieving more accurate results. Bear in mind that if we disregard variance as explained in Section 3.7, an MSRE difference of 0.0025 (marked with a dashed line in the bar graphs of Figure 5.14) for a particular sample analysis set translates to a five percent difference in the MRE for a given set. The *TMCBN* improvement displayed in these bar graphs appears to be substantial for many sample size nine, six and three cases. Based on these results, we join Tongue in urging that modified versions of *Rayleigh, Grubbs,* and *MRand* be empirically developed similar to *TMCBN.*

Tongue derived his *TMCBN* formula from a simulation design in which the maximum bias considered was 2.0σ. In Figure 5.14 the sample analysis sets labeled 80-100 correspond to those with a sample bias of greater than $2.75\,\overline{\sigma}$; note that for these sets *TMCBN* usually has a higher MSRE than *Numerical.* Like *MRand,* it appears that *TMCBN* has exclusionary bounds based on the region considered during its development. In addition to developing new modifications for *Rayleigh, Grubbs,* and *MRand,* Tongue's formula for modifying the *CBN* (*Numerical*) estimator could also be adjusted for use at higher bias levels.

2. Developing An Improved Decision Grid: Due to time constraints, we limited the number of levels for each factor to five or less. We recommend expanding the number of factor levels for the two most significant factors, bias and sample size, in a future study. Specifically, we feel that all sample sizes between three and ten and bias levels which are greater than 4.0σ should be explored. With the additional detail of such a design, such a

study would produce more detailed sample size/sample bias grids than those produced in our study.

Our use of ten replications was a significant improvement over the earlier CEP estimator comparison studies which used no replications. We recommend that future studies use even more replications to achieve increased confidence in the accuracy of the results.

As explained in Section 4.1, our *Numerical* and *TMCBN* results were distorted at sample size three. This distortion was due to certain design points and sample analysis sets using differing precision tolerances to enable the numerical root finding algorithm to converge. Perhaps one of these numerical CEP estimators is the estimator of choice at sample size three whenever the root finding algorithm converges at tolerance 0.01. Further study is required to determine the true performance of these two numerical estimators at this sample size.

Finally, we close by recommending that *MRand* be reevaluated in a study that excludes any design points outside *MRand*'s allowable boundary conditions. Because we included design points that were outside *MRand*'s feasible region, we could not fairly evaluate this estimator.

Appendix A: Notation Used in the Thesis

A.1) Population Parameters and Sample Statistics: This first section describes the notation used in this thesis for the population parameters and their corresponding sample statistics. The formulas for calculating the sample statistics is also presented.

n = The number of sample point values.

μ_x = The population crossrange mean.

$$\bar{x} = \text{The sample crossrange mean} = \frac{1}{n}\sum_{i=1}^{n} x_i . \tag{A-1}$$

μ_y = The population downrange mean.

$$\bar{y} = \text{The sample downrange mean} = \frac{1}{n}\sum_{i=1}^{n} y_i . \tag{A-2}$$

σ_x = The population crossrange standard deviation.

$$s_x = \text{The sample crossrange standard deviation} = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}} . \tag{A-3}$$

σ_y = The population downrange standard deviation.

$$s_y = \text{The sample downrange standard deviation} = \sqrt{\frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n-1}} . \tag{A-4}$$

ρ = The population correlation between corresponding crossrange and downrange values.

$$\bar{\rho} = \text{The sample correlation} = \frac{\sum_{i=1}^{n}(x - \bar{x})(y - \bar{y})}{(n-1)\, S_x\, S_y} . \tag{A-5}$$

$$\text{bias} = \text{The population bias} = \sqrt{\mu_x^{\,2} + \mu_y^{\,2}} \tag{A-6}$$

$\hat{bias}$ = The sample bias = $\sqrt{\bar{x}^2 + \bar{y}^2}$. (A-7)

r_i = The radial miss distance for point $(x_i, y_i) = \sqrt{x_i^2 + y_i^2}$. (A-8)

$\bar{r}$ = The mean radial miss distance $= \frac{1}{n}\sum_{i=1}^{n} r_i$. (A-9)

$\hat{ellip}$ = The sample ellipticity = $\{ s_y / s_x \text{ if } s_x \geq s_y,\ \ s_x / s_y \text{ if } s_x < s_y \}$. (A-10)

Scaling factor $\sigma = \sqrt{\dfrac{\sigma_x^2 + \sigma_y^2}{2}}$. (A-11)

$\bar{\sigma}$ = A sample estimator for $\sigma = \sqrt{\dfrac{S_x^2 + S_y^2}{2}}$ (A-12)

In the above sample statistics, $\bar{x}$, $\bar{y}$, s_x, s_y, and $\bar{\rho}$ are sufficient, minimum variance, and unbiased estimators. In addition, $\bar{x}$ and $\bar{y}$ are maximum likelihood estimators [Graybill (1976), pages 343-351].

A.2) The Thesis CEP Estimators: In this section the notation for each of the eight CEP estimators compared in the thesis experiment is presented.

Smed = sample median CEP estimator.

Ethridge = Ethridge CEP estimator.

MRand = modified Rand R-234 CEP estimator.

Valstar = Valstar CEP estimator.

Grubbs = Grubbs-Patniak / chi-square CEP estimator.

Rayleigh = Rayleigh distribution CEP estimator.

Numerical = direct numerical integration CEP estimator.

TMCBN = Tongue's modified CBN CEP estimator.

A.3) Measures of Effectiveness: In this final section of Appendix A, the notation for the measures of effectiveness (MOEs) used in this thesis is described.

RE = Relative error = | actual value - estimate | / actual value. (A-13)

MRE = The mean relative error for a set of n estimates $= \frac{1}{n}\sum_{i=1}^{n} RE_i$. (A-14)

var (RE) = The variance in the relative error of a set of n estimates

$$= \frac{1}{n-1}\sum_{i=1}^{n}(RE_i - MRE)^2 . \quad \text{(A-15)}$$

MSRE = The mean square of the relative error for a set of n estimates

$$= \frac{1}{n}\sum_{i=1}^{n} RE_i^{\,2} . \quad \text{(A-16)}$$

Appendix B: The MathCAD Design Point Template

Modified versions of this template were used to find parameters for the design points so that each design point would have a CEP of 100. In the example displayed, the design points for bias = σ along an axis 30^0 above the X-axis are found.

$$\text{ORIGIN} := 1 \qquad \text{TOL} := .01$$

$$i := 1..25$$

$$\theta := \frac{\pi}{6}$$

$$b := 1$$

$$n := \begin{bmatrix} -.8 & .2 \\ -.8 & .6 \\ -.8 & 1 \\ -.8 & 1.667 \\ -.8 & 5 \\ -.4 & .2 \\ -.4 & .6 \\ -.4 & 1 \\ -.4 & 1.667 \\ -.4 & 5 \\ 0 & .2 \\ 0 & .6 \\ 0 & 1 \\ 0 & 1.667 \\ 0 & 5 \\ .4 & .2 \\ .4 & .6 \\ .4 & 1 \\ .4 & 1.667 \\ .4 & 5 \\ .8 & .2 \\ .8 & .6 \\ .8 & 1 \\ .8 & 1.667 \\ .8 & 5 \end{bmatrix}$$

$$\mu x(\sigma x) := \frac{b \cdot \sigma x \cdot \sqrt{\dfrac{1 + \left(n_{i,2}\right)^2}{2}}}{\sec(\theta)} \qquad \mu y(\sigma x) := \mu x(\sigma x) \cdot \tan(\theta)$$

$$\sigma y(\sigma x) := n_{i,2} \cdot \sigma x \qquad p_i := n_{i,1}$$

1.) First, an initial guess for σx is made using the Grubbs-Patniak / Wilson-Hilferty formula:

$$m(\sigma x) := \left[(\mu x(\sigma x))^2 + (\mu y(\sigma x))^2 + (\sigma x)^2 + (\sigma y(\sigma x))^2\right]$$

$$v2(\sigma x) := (\mu x(\sigma x))^2 \cdot \sigma x^2 + 2 \cdot \mu x(\sigma x) \cdot \mu y(\sigma x) \cdot p_i \cdot \sigma x \cdot (\sigma y(\sigma x)) + (\mu y(\sigma x))^2 \cdot (\sigma y(\sigma x))^2$$

$$v(\sigma x) := 2 \cdot \left[\sigma x^4 + 2 \cdot \left[\left(p_i\right)^2 \cdot (\sigma x)^2 \cdot (\sigma y(\sigma x))^2\right] + (\sigma y(\sigma x))^4\right] + 4 \cdot v2(\sigma x)$$

$$g(\sigma x) := \sqrt{m(\sigma x) \cdot \left(1 - \frac{v(\sigma x)}{9 \cdot m(\sigma x)^2}\right)^3}$$

$$\text{guess}_i := \left| \begin{array}{l} \text{for } \sigma x \in 0, 0.1 .. 10000 \\ \quad \text{break if } \left|(g(\sigma x) - 100)\right| \le 1 \\ \sigma x \text{ otherwise} \end{array} \right.$$

2.) Next, numerical approximation is used to find the approximation:

$$\psi(\sigma x,x,y) := \frac{1}{2\cdot\left[1-(p_i)^2\right]}\cdot\left[\left(\frac{x-\mu x(\sigma x)}{\sigma x}\right)^2 - 2\cdot p_i\cdot\left[\frac{(x-\mu x(\sigma x))\cdot(y-\mu y(\sigma x))}{\sigma x\cdot(\sigma y(\sigma x))}\right] + \left(\frac{y-\mu y(\sigma x)}{\sigma y(\sigma x)}\right)^2\right]$$

$$f(\sigma x,x,y) := \left[2\cdot\pi\cdot\sigma x\cdot(\sigma y(\sigma x))\cdot\sqrt{1-(p_i)^2}\right]^{-1}\cdot e^{-\psi(\sigma x,x,y)}$$

$$\sigma x_i := guess_i$$

$$c(i,\sigma x) := root\left[\int_{-100}^{100}\int_{-\sqrt{10000-y^2}}^{\sqrt{10000-y^2}} f(\sigma x,x,y)\,dx\,dy - .5, \sigma x\right] \qquad sigx_i := c(i,\sigma x_i)$$

3.) Last, we ensure that our approximated values result in a CEP of 100:

$$\psi(x,y) := \frac{1}{2\cdot\left[1-(p_i)^2\right]}\cdot\left[\left(\frac{x-\mu x(sigx_i)}{sigx_i}\right)^2 - 2\cdot p_i\cdot\left[\frac{(x-\mu x(sigx_i))\cdot(y-\mu y(sigx_i))}{sigx_i\cdot(\sigma y(sigx_i))}\right] + \left(\frac{y-\mu y(sigx_i)}{\sigma y(sigx_i)}\right)^2\right]$$

$$f(x,y) := \left[2\cdot\pi\cdot sigx_i\cdot(\sigma y(sigx_i))\cdot\sqrt{1-(p_i)^2}\right]^{-1}\cdot e^{-\psi(x,y)}$$

$$r_i := 100$$

$$pr(i,r) := root\left[\int_{-r}^{r}\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} f(x,y)\,dx\,dy - .5, r\right] \qquad CEP_i := pr(i,r_i)$$

$$\sigma_i := \sqrt{\frac{(sigx_i)^2 + (n_{i,2}\cdot sigx_i)^2}{2}}$$

$$ux_i := \frac{b\cdot sigx_i\cdot\sqrt{\frac{1+(n_{i,2})^2}{2}}}{\sec(\theta)} \qquad uy_i := \mu x(sigx_i)\cdot\tan(\theta)$$

$$bias_i := \sqrt{(ux_i)^2 + (uy_i)^2}$$

$v_{i,1} := \frac{ux_i}{\bullet_i}$ $\quad$ $v_{i,2} := \frac{uy_i}{\bullet_i}$ $\quad$ The first and second columns of v contain the values of μ_x and μ_y, scaled against σ.

$v_{i,3} := \frac{sigx_i}{\bullet_i}$ $\quad$ $v_{i,4} := \frac{n_{i,2} \cdot sigx_i}{\bullet_i}$ $\quad$ The third and fourth columns of v contain the values of σ_x and σ_y, scaled against σ.

$w_{i,1} := \frac{ux_i}{100}$ $\quad$ $w_{i,2} := \frac{uy_i}{100}$ $\quad$ The first and second columns of w contain the values of μ_x and μ_y, scaled against the CEP of 100.

$w_{i,3} := \frac{sigx_i}{100}$ $\quad$ $w_{i,4} := \frac{n_{i,2} \cdot sigx_i}{100}$ $\quad$ The third and fourth columns of w contain the values of σ_x and σ_y, scaled against the CEP of 100.

$b_{i,1} := \frac{bias_i}{\bullet_i}$ $\quad$ Column 1 of matrix b contains the bias values scaled against σ.

$b_{i,2} := \frac{bias_i}{100}$ $\quad$ Column 2 of matrix b contains the bias values scaled against the CEP of 100.

n $\quad$ Matrix n lists the corresponding correlation and σ_y/σ_x values

$input_{i,1} := p_i$ $\quad$ Column 1 of the "input" matrix contains the correlation values

$input_{i,2} := ux_i$ $\quad$ Column 2 of the "input" matrix contains the raw unscaled μ_x values

$input_{i,3} := uy_i$ $\quad$ Column 3 of the "input" matrix contains the raw unscaled μ_y values

$input_{i,4} := sigx_i$ $\quad$ Column 4 of the "input" matrix contains the raw unscaled σ_x values

$input_{i,5} := n_{i,2} \cdot sigx_i$ $\quad$ Column 5 of the "input" matrix contains the raw unscaled σ_y values

Finally, the CEP vector shows the numerically integrated CEP calculation for each of the sets of $\{\mu x, \mu y, \sigma x, \sigma y, \rho\}$ found in steps 1 and 2. Note how close each comes to the predetermined CEP value of 100.

$$v = \begin{bmatrix} 0.866 & 0.5 & 1.387 & 0.277 \\ 0.866 & 0.5 & 1.213 & 0.728 \\ 0.866 & 0.5 & 1 & 1 \\ 0.866 & 0.5 & 0.727 & 1.213 \\ 0.866 & 0.5 & 0.277 & 1.387 \\ 0.866 & 0.5 & 1.387 & 0.277 \\ 0.866 & 0.5 & 1.213 & 0.728 \\ 0.866 & 0.5 & 1 & 1 \\ 0.866 & 0.5 & 0.727 & 1.213 \\ 0.866 & 0.5 & 0.277 & 1.387 \\ 0.866 & 0.5 & 1.387 & 0.277 \\ 0.866 & 0.5 & 1.213 & 0.728 \\ 0.866 & 0.5 & 1 & 1 \\ 0.866 & 0.5 & 0.727 & 1.213 \\ 0.866 & 0.5 & 0.277 & 1.387 \\ 0.866 & 0.5 & 1.387 & 0.277 \\ 0.866 & 0.5 & 1.213 & 0.728 \\ 0.866 & 0.5 & 1 & 1 \\ 0.866 & 0.5 & 0.727 & 1.213 \\ 0.866 & 0.5 & 0.277 & 1.387 \\ 0.866 & 0.5 & 1.387 & 0.277 \\ 0.866 & 0.5 & 1.213 & 0.728 \\ 0.866 & 0.5 & 1 & 1 \\ 0.866 & 0.5 & 0.727 & 1.213 \\ 0.866 & 0.5 & 0.277 & 1.387 \end{bmatrix} \quad w = \begin{bmatrix} 0.674 & 0.389 & 1.079 & 0.216 \\ 0.61 & 0.352 & 0.853 & 0.512 \\ 0.584 & 0.337 & 0.675 & 0.675 \\ 0.584 & 0.337 & 0.491 & 0.818 \\ 0.628 & 0.363 & 0.201 & 1.006 \\ 0.672 & 0.388 & 1.076 & 0.215 \\ 0.598 & 0.345 & 0.838 & 0.503 \\ 0.574 & 0.331 & 0.662 & 0.662 \\ 0.572 & 0.331 & 0.481 & 0.802 \\ 0.62 & 0.358 & 0.199 & 0.993 \\ 0.677 & 0.391 & 1.084 & 0.217 \\ 0.611 & 0.353 & 0.856 & 0.513 \\ 0.586 & 0.339 & 0.677 & 0.677 \\ 0.582 & 0.336 & 0.489 & 0.816 \\ 0.623 & 0.36 & 0.2 & 0.998 \\ 0.687 & 0.397 & 1.1 & 0.22 \\ 0.641 & 0.37 & 0.897 & 0.538 \\ 0.616 & 0.355 & 0.711 & 0.711 \\ 0.607 & 0.35 & 0.51 & 0.85 \\ 0.635 & 0.367 & 0.203 & 1.017 \\ 0.7 & 0.404 & 1.12 & 0.224 \\ 0.69 & 0.398 & 0.966 & 0.58 \\ 0.673 & 0.388 & 0.777 & 0.777 \\ 0.656 & 0.379 & 0.551 & 0.919 \\ 0.654 & 0.378 & 0.21 & 1.048 \end{bmatrix} \quad n = \begin{bmatrix} -0.8 & 0.2 \\ -0.8 & 0.6 \\ -0.8 & 1 \\ -0.8 & 1.667 \\ -0.8 & 5 \\ -0.4 & 0.2 \\ -0.4 & 0.6 \\ -0.4 & 1 \\ -0.4 & 1.667 \\ -0.4 & 5 \\ 0 & 0.2 \\ 0 & 0.6 \\ 0 & 1 \\ 0 & 1.667 \\ 0 & 5 \\ 0.4 & 0.2 \\ 0.4 & 0.6 \\ 0.4 & 1 \\ 0.4 & 1.667 \\ 0.4 & 5 \\ 0.8 & 0.2 \\ 0.8 & 0.6 \\ 0.8 & 1 \\ 0.8 & 1.667 \\ 0.8 & 5 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 & 0.778 \\ 1 & 0.704 \\ 1 & 0.675 \\ 1 & 0.675 \\ 1 & 0.726 \\ 1 & 0.776 \\ 1 & 0.691 \\ 1 & 0.662 \\ 1 & 0.661 \\ 1 & 0.716 \\ 1 & 0.782 \\ 1 & 0.706 \\ 1 & 0.677 \\ 1 & 0.672 \\ 1 & 0.72 \\ 1 & 0.793 \\ 1 & 0.74 \\ 1 & 0.711 \\ 1 & 0.701 \\ 1 & 0.733 \\ 1 & 0.808 \\ 1 & 0.797 \\ 1 & 0.777 \\ 1 & 0.758 \\ 1 & 0.756 \end{bmatrix} \qquad CEP = \begin{bmatrix} 99.878 \\ 99.986 \\ 99.997 \\ 99.989 \\ 100.038 \\ 99.92 \\ 100 \\ 100.002 \\ 100.002 \\ 99.993 \\ 99.92 \\ 99.992 \\ 99.998 \\ 100.002 \\ 99.983 \\ 99.901 \\ 99.971 \\ 99.985 \\ 99.999 \\ 99.976 \\ 99.888 \\ 99.908 \\ 99.939 \\ 99.963 \\ 99.918 \end{bmatrix}$$

Appendix C: The Design Points for Each Sample Size

NOTATION KEY

Reference Number = reference number for the given design point
μ_x = crossrange mean value
μ_y = downrange mean value
CEP_{MPI} = the CEP about the mean point of impact
($*\sigma$) = multiple of the σ value

$\theta = \tan^{-1}(\mu_x/\mu_y)$
σ_x = crossrange standard deviation
σ_y = crossrange standard deviation
ρ = correlation
(*CEP) = multiple of the CEP value

Reference Number	θ	μ_x ($*\sigma$)	μ_y ($*\sigma$)	σ_x ($*\sigma$)	σ_y ($*\sigma$)	μ_x (*CEP)	μ_y (*CEP)	σ_x (*CEP)	σ_y (*CEP)	ρ	σ_y/σ_x	bias ($*\sigma$)	bias (*CEP)	CEP_{MPI} (*CEP)	CEP/CEP_{MPI}
1	0	0	0	1.387	0.277	0	0	1.416	0.283	0	0.2	0	0	1	1
2	0	0	0	1.213	0.728	0	0	1.07	0.642	0	0.6	0	0	1	1
3	0	0	0	1	1	0	0	0.848	0.848	0	1	0	0	1	1
4	0	0	0	0.727	1.213	0	0	0.642	1.07	0	1.667	0	0	1	1
5	0	0	0	0.277	1.387	0	0	0.283	1.414	0	5	0	0	1	1
6	0	0	0	1.387	0.277	0	0	1.423	0.285	0.4	0.2	0	0	1	1
7	0	0	0	1.213	0.728	0	0	1.098	0.659	0.4	0.6	0	0	1	1
8	0	0	0	1	1	0	0	0.872	0.872	0.4	1	0	0	1	1
9	0	0	0	0.727	1.213	0	0	0.658	1.097	0.4	1.667	0	0	1	1
10	0	0	0	0.277	1.387	0	0	0.284	1.421	0.4	5	0	0	1	1
11	0	0	0	1.387	0.277	0	0	1.441	0.288	0.8	0.2	0	0	1	1
12	0	0	0	1.213	0.728	0	0	1.2	0.72	0.8	0.6	0	0	1	1
13	0	0	0	1	1	0	0	0.969	0.969	0.8	1	0	0	1	1
14	0	0	0	0.727	1.213	0	0	0.719	1.199	0.8	1.667	0	0	1	1
15	0	0	0	0.277	1.387	0	0	0.288	1.439	0.8	5	0	0	1	1
16	0	0.5	0	1.387	0.277	0.481	0	1.335	0.267	0	0.2	0.5	0.481	.94234	1.061
17	0	0.5	0	1.213	0.728	0.42	0	1.018	0.611	0	0.6	0.5	0.42	.95093	1.052
18	0	0.5	0	1	1	0.399	0	0.798	0.798	0	1	0.5	0.399	.94074	1.063

Reference Number	θ	μ_x (*σ)	μ_y (*σ)	σ_x (*σ)	σ_y (*σ)	μ_x (*CEP)	μ_y (*CEP)	σ_x (*CEP)	σ_y (*CEP)	ρ	σ_y/σ_x	bias (*σ)	bias (*CEP)	CEP_{MPI} (*CEP)	CEP/CEP_{MPI}
19	0	0.5	0	0.727	1.213	0.404	0	0.587	0.979	0	1.667	0.5	0.404	.91498	1.093
20	0	0.5	0	0.277	1.387	0.447	0	0.248	1.239	0	5	0.5	0.447	.87533	1.142
21	0	0.5	0	1.387	0.277	0.483	0	1.34	0.268	0.4	0.2	0.5	0.483	.94125	1.062
22	0	0.5	0	1.213	0.728	0.428	0	1.039	0.623	0.4	0.6	0.5	0.428	.94626	1.057
23	0	0.5	0	1	1	0.408	0	0.815	0.815	0.4	1	0.5	0.408	.93479	1.07
24	0	0.5	0	0.727	1.213	0.411	0	0.598	0.996	0.4	1.667	0.5	0.411	.90829	1.101
25	0	0.5	0	0.277	1.387	0.45	0	0.249	1.247	0.4	5	0.5	0.45	.8774	1.14
26	0	0.5	0	1.387	0.277	0.488	0	1.354	0.271	0.8	0.2	0.5	0.488	.93875	1.065
27	0	0.5	0	1.213	0.728	0.46	0	1.115	0.669	0.8	0.6	0.5	0.46	.92836	1.077
28	0	0.5	0	1	1	0.442	0	0.883	0.883	0.8	1	0.5	0.442	.9108	1.098
29	0	0.5	0	0.727	1.213	0.439	0	0.639	1.066	0.8	1.667	0.5	0.439	.88863	1.125
30	0	0.5	0	0.277	1.387	0.459	0	0.255	1.273	0.8	5	0.5	0.459	.88456	1.131
31	30	0.433	0.25	1.387	0.277	0.413	0.238	1.321	0.264	-0.8	0.2	0.5	0.476	.91633	1.091
32	30	0.433	0.25	1.213	0.728	0.382	0.221	1.071	0.643	-0.8	0.6	0.5	0.442	.89202	1.121
33	30	0.433	0.25	1	1	0.371	0.214	0.856	0.856	-0.8	1	0.5	0.428	.8826	1.133
34	30	0.433	0.25	0.727	1.213	0.374	0.216	0.629	1.049	-0.8	1.667	0.5	0.432	.87425	1.144
35	30	0.433	0.25	0.277	1.387	0.4	0.231	0.256	1.281	-0.8	5	0.5	0.462	.89006	1.124
36	30	0.433	0.25	1.387	0.277	0.409	0.236	1.31	0.262	-0.4	0.2	0.5	0.472	.91978	1.087
37	30	0.433	0.25	1.213	0.728	0.362	0.209	1.014	0.608	-0.4	0.6	0.5	0.418	.92343	1.083
38	30	0.433	0.25	1	1	0.348	0.201	0.804	0.804	-0.4	1	0.5	0.402	.92218	1.084
39	30	0.433	0.25	0.727	1.213	0.355	0.205	0.596	0.993	-0.4	1.667	0.5	0.41	.90562	1.104
40	30	0.433	0.25	0.277	1.387	0.394	0.228	0.253	1.263	-0.4	5	0.5	0.455	.88816	1.126
41	30	0.433	0.25	1.387	0.277	0.409	0.236	1.31	0.262	0	0.2	0.5	0.472	.92463	1.082
42	30	0.433	0.25	1.213	0.728	0.36	0.208	1.008	0.605	0	0.6	0.5	0.415	.94131	1.062
43	30	0.433	0.25	1	1	0.346	0.2	0.798	0.798	0	1	0.5	0.399	.94071	1.063
44	30	0.433	0.25	0.727	1.213	0.353	0.204	0.593	0.988	0	1.667	0.5	0.407	.92343	1.083
45	30	0.433	0.25	0.277	1.387	0.394	0.227	0.252	1.261	0	5	0.5	0.455	.89116	1.122

Reference Number	θ	μ_x (*σ)	μ_y (*σ)	σ_x (*σ)	σ_y (*σ)	μ_x (*CEP)	μ_y (*CEP)	σ_x (*CEP)	σ_y (*CEP)	ρ	σ_y/σ_x	bias (*σ)	bias (*CEP)	CEP_{MPI} (*CEP)	CEP/CEP_{MPI}
46	30	0.433	0.25	1.387	0.277	0.413	0.239	1.323	0.265	0.4	0.2	0.5	0.477	.92936	1.076
47	30	0.433	0.25	1.213	0.728	0.372	0.215	1.042	0.625	0.4	0.6	0.5	0.43	.94936	1.053
48	30	0.433	0.25	1	1	0.358	0.207	0.827	0.827	0.4	1	0.5	0.413	.94802	1.055
49	30	0.433	0.25	0.727	1.213	0.364	0.21	0.611	1.019	0.4	1.667	0.5	0.42	.92933	1.076
50	30	0.433	0.25	0.277	1.387	0.398	0.23	0.255	1.276	0.4	5	0.5	0.46	.89723	1.115
51	30	0.433	0.25	1.387	0.277	0.42	0.242	1.345	0.269	0.8	0.2	0.5	0.485	.93267	1.072
52	30	0.433	0.25	1.213	0.728	0.406	0.234	1.136	0.682	0.8	0.6	0.5	0.469	.94638	1.057
53	30	0.433	0.25	1	1	0.396	0.229	0.914	0.914	0.8	1	0.5	0.457	.94314	1.06
54	30	0.433	0.25	0.727	1.213	0.396	0.229	0.665	1.109	0.8	1.667	0.5	0.457	.92428	1.082
55	30	0.433	0.25	0.277	1.387	0.407	0.235	0.261	1.303	0.8	5	0.5	0.47	.90512	1.105
56	60	0.25	0.433	1.387	0.277	0.231	0.4	1.282	0.256	-0.8	0.2	0.5	0.462	.88884	1.125
57	60	0.25	0.433	1.213	0.728	0.216	0.375	1.049	0.63	-0.8	0.6	0.5	0.433	.87394	1.144
58	60	0.25	0.433	1	1	0.214	0.371	0.856	0.856	-0.8	1	0.5	0.428	.8825	1.133
59	60	0.25	0.433	0.727	1.213	0.221	0.382	0.642	1.07	-0.8	1.667	0.5	0.441	.89204	1.121
60	60	0.25	0.433	0.277	1.387	0.238	0.411	0.264	1.318	-0.8	5	0.5	0.475	.91523	1.093
61	60	0.25	0.433	1.387	0.277	0.228	0.395	1.264	0.253	-0.4	0.2	0.5	0.456	.88754	1.127
62	60	0.25	0.433	1.213	0.728	0.205	0.355	0.994	0.596	-0.4	0.6	0.5	0.41	.9053	1.105
63	60	0.25	0.433	1	1	0.201	0.348	0.804	0.804	-0.4	1	0.5	0.402	.92209	1.084
64	60	0.25	0.433	0.727	1.213	0.209	0.362	0.608	1.013	-0.4	1.667	0.5	0.418	.92349	1.083
65	60	0.25	0.433	0.277	1.387	0.236	0.408	0.261	1.306	-0.4	5	0.5	0.471	.9189	1.088
66	60	0.25	0.433	1.387	0.277	0.228	0.394	1.262	0.252	0	0.2	0.5	0.455	.89057	1.123
67	60	0.25	0.433	1.213	0.728	0.204	0.353	0.988	0.593	0	0.6	0.5	0.407	.92318	1.083
68	60	0.25	0.433	1	1	0.2	0.346	0.798	0.798	0	1	0.5	0.399	.94061	1.063
69	60	0.25	0.433	0.727	1.213	0.208	0.36	0.604	1.007	0	1.667	0.5	0.415	.94132	1.062
70	60	0.25	0.433	0.277	1.387	0.236	0.408	0.261	1.307	0	5	0.5	0.471	.92362	1.083
71	60	0.25	0.433	1.387	0.277	0.23	0.399	1.277	0.255	0.4	0.2	0.5	0.46	.89664	1.115
72	60	0.25	0.433	1.213	0.728	0.21	0.364	1.02	0.612	0.4	0.6	0.5	0.42	.92908	1.076

Reference Number	θ	μ_x (*σ)	μ_y (*σ)	σ_x (*σ)	σ_y (*σ)	μ_x (*CEP)	μ_y (*CEP)	σ_x (*CEP)	σ_y (*CEP)	ρ	σ_y/σ_x	bias (*σ)	bias (*CEP)	CEP_{MPI} (*CEP)	CEP/CEP_{MPI}
73	60	0.25	0.433	1	1	0.207	0.358	0.827	0.827	0.4	1	0.5	0.413	.94793	1.055
74	60	0.25	0.433	0.727	1.213	0.215	0.372	0.625	1.041	0.4	1.667	0.5	0.429	.94934	1.053
75	60	0.25	0.433	0.277	1.387	0.238	0.412	0.264	1.32	0.4	5	0.5	0.476	.92817	1.077
76	60	0.25	0.433	1.387	0.277	0.235	0.407	1.304	0.261	0.8	0.2	0.5	0.47	.90459	1.105
77	60	0.25	0.433	1.213	0.728	0.229	0.396	1.109	0.666	0.8	0.6	0.5	0.457	.92395	1.082
78	60	0.25	0.433	1	1	0.229	0.396	0.914	0.914	0.8	1	0.5	0.457	.94302	1.06
79	60	0.25	0.433	0.727	1.213	0.234	0.405	0.681	1.135	0.8	1.667	0.5	0.468	.94645	1.057
80	60	0.25	0.433	0.277	1.387	0.242	0.42	0.269	1.343	0.8	5	0.5	0.484	.93325	1.072
81	0	1	0	1.387	0.277	0.812	0	1.126	0.225	0	0.2	1	0.812	.79467	1.258
82	0	1	0	1.213	0.728	0.725	0	0.88	0.528	0	0.6	1	0.725	.82174	1.217
83	0	1	0	1	1	0.677	0	0.677	0.677	0	1	1	0.677	.79825	1.253
84	0	1	0	0.727	1.213	0.658	0	0.479	0.798	0	1.667	1	0.658	.74642	1.34
85	0	1	0	0.277	1.387	0.691	0	0.192	0.959	0	5	1	0.691	.67743	1.476
86	0	1	0	1.387	0.277	0.814	0	1.128	0.226	0.4	0.2	1	0.814	.79242	1.262
87	0	1	0	1.213	0.728	0.733	0	0.889	0.533	0.4	0.6	1	0.733	.81012	1.234
88	0	1	0	1	1	0.685	0	0.685	0.685	0.4	1	1	0.685	.78519	1.274
89	0	1	0	0.727	1.213	0.665	0	0.484	0.807	0.4	1.667	1	0.665	.73561	1.359
90	0	1	0	0.277	1.387	0.696	0	0.193	0.965	0.4	5	1	0.696	.67904	1.473
91	0	1	0	1.387	0.277	0.818	0	1.135	0.227	0.8	0.2	1	0.818	.78703	1.271
92	0	1	0	1.213	0.728	0.766	0	0.929	0.557	0.8	0.6	1	0.766	.77351	1.293
93	0	1	0	1	1	0.72	0	0.72	0.72	0.8	1	1	0.72	.74265	1.347
94	0	1	0	0.727	1.213	0.698	0	0.508	0.846	0.8	1.667	1	0.698	.70546	1.418
95	0	1	0	0.277	1.387	0.715	0	0.198	0.991	0.8	5	1	0.715	.68842	1.453
96	30	0.866	0.5	1.387	0.277	0.674	0.389	1.079	0.216	-0.8	0.2	1	0.778	.74822	1.337
97	30	0.866	0.5	1.213	0.728	0.61	0.352	0.853	0.512	-0.8	0.6	1	0.704	.71084	1.407
98	30	0.866	0.5	1	1	0.584	0.337	0.675	0.675	-0.8	1	1	0.675	.69587	1.437
99	30	0.866	0.5	0.727	1.213	0.584	0.337	0.491	0.818	-0.8	1.667	1	0.675	.68229	1.466

Reference Number	θ	μ_x (*σ)	μ_y (*σ)	σ_x (*σ)	σ_y (*σ)	μ_x (*CEP)	μ_y (*CEP)	σ_x (*CEP)	σ_y (*CEP)	ρ	σ_y/σ_x	bias (*σ)	bias (*CEP)	CEP_{MPI} (*CEP)	CEP/CEP_{MPI}
100	30	0.866	0.5	0.277	1.387	0.628	0.363	0.201	1.006	-0.8	5	1	0.726	69891	1.431
101	30	0.866	0.5	1.387	0.277	0.672	0.388	1.076	0.215	-0.4	0.2	1	0.776	.75596	1.323
102	30	0.866	0.5	1.213	0.728	0.598	0.345	0.838	0.503	-0.4	0.6	1	0.691	.76306	1.311
103	30	0.866	0.5	1	1	0.574	0.331	0.662	0.662	-0.4	1	1	0.662	.75942	1.317
104	30	0.866	0.5	0.727	1.213	0.572	0.331	0.481	0.802	-0.4	1.667	1	0.661	.73092	1.368
105	30	0.866	0.5	0.277	1.387	0.62	0.358	0.199	0.993	-0.4	5	1	0.716	.69869	1.431
106	30	0.866	0.5	1.387	0.277	0.677	0.391	1.084	0.217	0	0.2	1	0.782	.76505	1.307
107	30	0.866	0.5	1.213	0.728	0.611	0.353	0.856	0.513	0	0.6	1	0.706	.79938	1.251
108	30	0.866	0.5	1	1	0.586	0.339	0.677	0.677	0	1	1	0.677	.79812	1.253
109	30	0.866	0.5	0.727	1.213	0.582	0.336	0.489	0.816	0	1.667	1	0.672	.76239	1.312
110	30	0.866	0.5	0.277	1.387	0.623	0.36	0.2	0.998	0	5	1	0.72	.70552	1.417
111	30	0.866	0.5	1.387	0.277	0.687	0.397	1.1	0.22	0.4	0.2	1	0.793	.77263	1.294
112	30	0.866	0.5	1.213	0.728	0.641	0.37	0.897	0.538	0.4	0.6	1	0.74	.81717	1.224
113	30	0.866	0.5	1	1	0.616	0.355	0.711	0.711	0.4	1	1	0.711	.81506	1.227
114	30	0.866	0.5	0.727	1.213	0.607	0.35	0.51	0.85	0.4	1.667	1	0.701	.77473	1.291
115	30	0.866	0.5	0.277	1.387	0.635	0.367	0.203	1.017	0.4	5	1	0.733	.71526	1.398
116	30	0.866	0.5	1.387	0.277	0.7	0.404	1.12	0.224	0.8	0.2	1	0.808	.77682	1.287
117	30	0.866	0.5	1.213	0.728	0.69	0.398	0.966	0.58	0.8	0.6	1	0.797	.80495	1.242
118	30	0.866	0.5	1	1	0.673	0.388	0.777	0.777	0.8	1	1	0.777	.80097	1.248
119	30	0.866	0.5	0.727	1.213	0.656	0.379	0.551	0.919	0.8	1.667	1	0.758	.76635	1.305
120	30	0.866	0.5	0.277	1.387	0.654	0.378	0.21	1.048	0.8	5	1	0.756	.72796	1.374
121	60	0.5	0.866	1.387	0.277	0.363	0.628	1.006	0.201	-0.8	0.2	1	0.725	.69767	1.433
122	60	0.5	0.866	1.213	0.728	0.337	0.584	0.818	0.491	-0.8	0.6	1	0.675	.68135	1.468
123	60	0.5	0.866	1	1	0.337	0.584	0.674	0.674	-0.8	1	1	0.674	.69559	1.438
124	60	0.5	0.866	0.727	1.213	0.352	0.609	0.512	0.853	-0.8	1.667	1	0.703	.71101	1.406
125	60	0.5	0.866	0.277	1.387	0.389	0.674	0.216	1.079	-0.8	5	1	0.778	.74978	1.334
126	60	0.5	0.866	1.387	0.277	0.358	0.62	0.993	0.199	-0.4	0.2	1	0.716	.69753	1.434

Reference Number	θ	μ_x (*σ)	μ_y (*σ)	σ_x (*σ)	σ_y (*σ)	μ_x (*CEP)	μ_y (*CEP)	σ_x (*CEP)	σ_y (*CEP)	ρ	σ_y/σ_x	bias (*σ)	bias (*CEP)	CEP_{MPI} (*CEP)	CEP/CEP_{MPI}
127	60	0.5	0.866	1.213	0.728	0.33	0.572	0.801	0.481	-0.4	0.6	1	0.661	.73011	1.37
128	60	0.5	0.866	1	1	0.331	0.573	0.662	0.662	-0.4	1	1	0.662	.75912	1.317
129	60	0.5	0.866	0.727	1.213	0.345	0.598	0.502	0.837	-0.4	1.667	1	0.69	.76319	1.31
130	60	0.5	0.866	0.277	1.387	0.388	0.672	0.215	1.076	-0.4	5	1	0.776	.7568	1.321
131	60	0.5	0.866	1.387	0.277	0.36	0.623	0.998	0.2	0	0.2	1	0.72	.70439	1.42
132	60	0.5	0.866	1.213	0.728	0.336	0.582	0.815	0.489	0	0.6	1	0.672	.76164	1.313
133	60	0.5	0.866	1	1	0.338	0.586	0.677	0.677	0	1	1	0.677	.79781	1.253
134	60	0.5	0.866	0.727	1.213	0.353	0.611	0.513	0.855	0	1.667	1	0.705	.79938	1.251
135	60	0.5	0.866	0.277	1.387	0.391	0.677	0.217	1.084	0	5	1	0.781	.76573	1.306
136	60	0.5	0.866	1.387	0.277	0.367	0.635	1.017	0.203	0.4	0.2	1	0.733	.71411	1.4
137	60	0.5	0.866	1.213	0.728	0.35	0.607	0.849	0.51	0.4	0.6	1	0.7	.77394	1.292
138	60	0.5	0.866	1	1	0.355	0.615	0.711	0.711	0.4	1	1	0.711	.81471	1.227
139	60	0.5	0.866	0.727	1.213	0.369	0.64	0.538	0.896	0.4	1.667	1	0.739	.81707	1.224
140	60	0.5	0.866	0.277	1.387	0.396	0.687	0.22	1.099	0.4	5	1	0.793	.77333	1.293
141	60	0.5	0.866	1.387	0.277	0.378	0.655	1.048	0.21	0.8	0.2	1	0.756	.72691	1.376
142	60	0.5	0.866	1.213	0.728	0.379	0.657	0.919	0.552	0.8	0.6	1	0.758	.76566	1.306
143	60	0.5	0.866	1	1	0.388	0.672	0.776	0.776	0.8	1	1	0.776	.80061	1.249
144	60	0.5	0.866	0.727	1.213	0.398	0.69	0.579	0.966	0.8	1.667	1	0.796	.80514	1.242
145	60	0.5	0.866	0.277	1.387	0.403	0.698	0.224	1.118	0.8	5	1	0.806	.7769	1.287
146	0	2	0	1.387	0.277	0.987	0	0.684	0.137	0	0.2	2	0.987	.48302	2.07
147	0	2	0	1.213	0.728	0.935	0	0.567	0.34	0	0.6	2	0.935	.52987	1.887
148	0	2	0	1	1	0.891	0	0.445	0.445	0	1	2	0.891	.52474	1.906
149	0	2	0	0.727	1.213	0.863	0	0.314	0.523	0	1.667	2	0.863	.48924	2.044
150	0	2	0	0.277	1.387	0.871	0	0.121	0.604	0	5	2	0.871	.42702	2.342
151	0	2	0	1.387	0.277	0.988	0	0.685	0.137	0.4	0.2	2	0.988	.48136	2.077
152	0	2	0	1.213	0.728	0.942	0	0.571	0.343	0.4	0.6	2	0.942	.52053	1.921
153	0	2	0	1	1	0.898	0	0.449	0.449	0.4	1	2	0.898	.51467	1.943

Reference Number	θ	μ_x (*σ)	μ_y (*σ)	σ_x (*σ)	σ_y (*σ)	μ_x (*CEP)	μ_y (*CEP)	σ_x (*CEP)	σ_y (*CEP)	ρ	σ_y/σ_x	bias (*σ)	bias (*CEP)	CEP_{MPI} (*CEP)	CEP/CEP_{MPI}
154	0	2	0	0.727	1.213	0.869	0	0.316	0.527	0.4	1.667	2	0.869	.48071	2.08
155	0	2	0	0.277	1.387	0.875	0	0.121	0.607	0.4	5	2	0.875	.42697	2.342
156	0	2	0	1.387	0.277	0.992	0	0.688	0.138	0.8	0.2	2	0.992	.47717	2.096
157	0	2	0	1.213	0.728	0.968	0	0.587	0.352	0.8	0.6	2	0.968	.48863	2.047
158	0	2	0	1	1	0.931	0	0.465	0.465	0.8	1	2	0.931	.48001	2.083
159	0	2	0	0.727	1.213	0.898	0	0.327	0.545	0.8	1.667	2	0.898	.45411	2.202
160	0	2	0	0.277	1.387	0.893	0	0.124	0.619	0.8	5	2	0.893	.42995	2.326
161	30	1.732	1	1.387	0.277	0.845	0.488	0.677	0.135	-0.8	0.2	2	0.976	.46932	2.131
162	30	1.732	1	1.213	0.728	0.785	0.453	0.55	0.33	-0.8	0.6	2	0.907	.45793	2.184
163	30	1.732	1	1	1	0.753	0.435	0.435	0.435	-0.8	1	2	0.87	.4484	2.23
164	30	1.732	1	0.727	1.213	0.747	0.432	0.314	0.523	-0.8	1.667	2	0.863	.43634	2.292
165	30	1.732	1	0.277	1.387	0.791	0.457	0.127	0.633	-0.8	5	2	0.913	.43984	2.274
166	30	1.732	1	1.387	0.277	0.844	0.487	0.676	0.135	-0.4	0.2	2	0.975	.47463	2.107
167	30	1.732	1	1.213	0.728	0.781	0.451	0.547	0.328	-0.4	0.6	2	0.902	.49846	2.006
168	30	1.732	1	1	1	0.753	0.435	0.435	0.435	-0.4	1	2	0.87	.49857	2.006
169	30	1.732	1	0.727	1.213	0.748	0.432	0.314	0.524	-0.4	1.667	2	0.864	.47745	2.094
170	30	1.732	1	0.277	1.387	0.788	0.455	0.126	0.631	-0.4	5	2	0.91	.44375	2.254
171	30	1.732	1	1.387	0.277	0.846	0.489	0.678	0.136	0	0.2	2	0.977	.47819	2.091
172	30	1.732	1	1.213	0.728	0.796	0.46	0.557	0.334	0	0.6	2	0.919	.52065	1.921
173	30	1.732	1	1	1	0.771	0.445	0.445	0.445	0	1	2	0.89	.52463	1.906
174	30	1.732	1	0.727	1.213	0.764	0.441	0.321	0.535	0	1.667	2	0.882	.50004	2
175	30	1.732	1	0.277	1.387	0.795	0.459	0.127	0.637	0	5	2	0.918	.45007	2.222
176	30	1.732	1	1.387	0.277	0.851	0.491	0.681	0.136	0.4	0.2	2	0.983	.47846	2.09
177	30	1.732	1	1.213	0.728	0.819	0.473	0.574	0.344	0.4	0.6	2	0.946	.52269	1.913
178	30	1.732	1	1	1	0.8	0.462	0.462	0.462	0.4	1	2	0.924	.52966	1.888
179	30	1.732	1	0.727	1.213	0.792	0.457	0.332	0.554	0.4	1.667	2	0.914	.50535	1.979
180	30	1.732	1	0.277	1.387	0.81	0.468	0.13	0.649	0.4	5	2	0.935	.45618	2.192

Reference Number	θ	μ_x (*σ)	μ_y (*σ)	σ_x (*σ)	σ_y (*σ)	μ_x (*CEP)	μ_y (*CEP)	σ_x (*CEP)	σ_y (*CEP)	ρ	σ_y/σ_x	bias (*σ)	bias (*CEP)	CEP_{MPI} (*CEP)	CEP/CEP_{MPI}
181	30	1.732	1	1.387	0.277	0.857	0.495	0.686	0.137	0.8	0.2	2	0.99	.47584	2.102
182	30	1.732	1	1.213	0.728	0.847	0.489	0.593	0.356	0.8	0.6	2	0.979	.49417	2.024
183	30	1.732	1	1	1	0.84	0.485	0.485	0.485	0.8	1	2	0.97	.50027	1.999
184	30	1.732	1	0.727	1.213	0.834	0.482	0.35	0.584	0.8	1.667	2	0.963	.48697	2.054
185	30	1.732	1	0.277	1.387	0.832	0.48	0.133	0.666	0.8	5	2	0.961	.4628	2.161
186	60	1	1.732	1.387	0.277	0.456	0.789	0.632	0.126	-0.8	0.2	2	0.911	.43807	2.283
187	60	1	1.732	1.213	0.728	0.431	0.747	0.523	0.314	-0.8	0.6	2	0.863	.43559	2.296
188	60	1	1.732	1	1	0.435	0.753	0.435	0.435	-0.8	1	2	0.87	.44842	2.23
189	60	1	1.732	0.727	1.213	0.453	0.785	0.33	0.55	-0.8	1.667	2	0.907	.45835	2.182
190	60	1	1.732	0.277	1.387	0.487	0.844	0.135	0.676	-0.8	5	2	0.975	.46938	2.13
191	60	1	1.732	1.387	0.277	0.455	0.788	0.631	0.126	-0.4	0.2	2	0.91	.44315	2.257
192	60	1	1.732	1.213	0.728	0.431	0.747	0.523	0.314	-0.4	0.6	2	0.862	.47641	2.099
193	60	1	1.732	1	1	0.434	0.753	0.434	0.434	-0.4	1	2	0.869	.49805	2.008
194	60	1	1.732	0.727	1.213	0.451	0.781	0.328	0.547	-0.4	1.667	2	0.902	.49858	2.006
195	60	1	1.732	0.277	1.387	0.487	0.844	0.135	0.676	-0.4	5	2	0.975	.47533	2.104
196	60	1	1.732	1.387	0.277	0.459	0.795	0.637	0.127	0	0.2	2	0.918	.44937	2.225
197	60	1	1.732	1.213	0.728	0.44	0.763	0.534	0.321	0	0.6	2	0.881	.49904	2.004
198	60	1	1.732	1	1	0.445	0.77	0.445	0.445	0	1	2	0.889	.52405	1.908
199	60	1	1.732	0.727	1.213	0.459	0.795	0.334	0.557	0	1.667	2	0.918	.52064	1.921
200	60	1	1.732	0.277	1.387	0.489	0.846	0.136	0.678	0	5	2	0.977	.47896	2.088
201	60	1	1.732	1.387	0.277	0.468	0.81	0.649	0.13	0.4	0.2	2	0.935	.45549	2.195
202	60	1	1.732	1.213	0.728	0.457	0.791	0.554	0.332	0.4	0.6	2	0.913	.50437	1.983
203	60	1	1.732	1	1	0.462	0.799	0.462	0.462	0.4	1	2	0.923	.52907	1.89
204	60	1	1.732	0.727	1.213	0.473	0.819	0.344	0.573	0.4	1.667	2	0.945	.52271	1.913
205	60	1	1.732	0.277	1.387	0.491	0.851	0.136	0.681	0.4	5	2	0.983	.4792	2.087
206	60	1	1.732	1.387	0.277	0.479	0.83	0.664	0.133	0.8	0.2	2	0.958	.46081	2.17
207	60	1	1.732	1.213	0.728	0.482	0.834	0.584	0.35	0.8	0.6	2	0.963	.48641	2.056

Reference Number	θ	μ_x (*σ)	μ_y (*σ)	σ_x (*σ)	σ_y (*σ)	μ_x (*CEP)	μ_y (*CEP)	σ_x (*CEP)	σ_y (*CEP)	ρ	σ_y/σ_x	bias (*σ)	bias (*CEP)	CEP_{MPI} (*CEP)	CEP/CEP_{MPI}
208	60	1	1.732	1	1	0.485	0.84	0.485	0.485	0.8	1	2	0.97	.49996	2
209	60	1	1.732	0.727	1.213	0.489	0.847	0.356	0.593	0.8	1.667	2	0.979	.4947	2.021
210	60	1	1.732	0.277	1.387	0.494	0.856	0.137	0.686	0.8	5	2	0.989	.4762	2.1
211	0	4	0	1.387	0.277	0.998	0	0.346	0.069	0	0.2	4	0.998	.24406	4.097
212	0	4	0	1.213	0.728	0.984	0	0.298	0.179	0	0.6	4	0.984	.27859	3.59
213	0	4	0	1	1	0.97	0	0.242	0.242	0	1	4	0.97	.28574	3.5
214	0	4	0	0.727	1.213	0.958	0	0.174	0.29	0	1.667	4	0.958	.27157	3.682
215	0	4	0	0.277	1.387	0.958	0	0.066	0.332	0	5	4	0.958	.23462	4.262
216	0	4	0	1.387	0.277	0.998	0	0.346	0.069	0.4	0.2	4	0.998	.24299	4.115
217	0	4	0	1.213	0.728	0.986	0	0.299	0.179	0.4	0.6	4	0.986	.2724	3.671
218	0	4	0	1	1	0.974	0	0.243	0.243	0.4	1	4	0.974	.27911	3.583
219	0	4	0	0.727	1.213	0.962	0	0.175	0.292	0.4	1.667	4	0.962	.266	3.759
220	0	4	0	0.277	1.387	0.96	0	0.067	0.333	0.4	5	4	0.96	.23413	4.271
221	0	4	0	1.387	0.277	0.999	0	0.346	0.069	0.8	0.2	4	0.999	.24022	4.163
222	0	4	0	1.213	0.728	0.994	0	0.301	0.181	0.8	0.6	4	0.994	.25099	3.984
223	0	4	0	1	1	0.988	0	0.247	0.247	0.8	1	4	0.988	.25479	3.925
224	0	4	0	0.727	1.213	0.979	0	0.178	0.297	0.8	1.667	4	0.979	.24753	4.04
225	0	4	0	0.277	1.387	0.971	0	0.067	0.337	0.8	5	4	0.971	.23382	4.277
226	30	3.464	2	1.387	0.277	0.865	0.499	0.346	0.069	-0.8	0.2	4	0.998	.24006	4.166
227	30	3.464	2	1.213	0.728	0.851	0.492	0.298	0.179	-0.8	0.6	4	0.983	.24826	4.028
228	30	3.464	2	1	1	0.833	0.481	0.24	0.24	-0.8	1	4	0.962	.24795	4.033
229	30	3.464	2	0.727	1.213	0.827	0.477	0.174	0.289	-0.8	1.667	4	0.954	.24124	4.145
230	30	3.464	2	0.277	1.387	0.861	0.497	0.069	0.345	-0.8	5	4	0.994	.23944	4.176
231	30	3.464	2	1.387	0.277	0.863	0.498	0.346	0.069	-0.4	0.2	4	0.997	.24276	4.119
232	30	3.464	2	1.213	0.728	0.846	0.489	0.296	0.178	-0.4	0.6	4	0.977	.26995	3.704
233	30	3.464	2	1	1	0.833	0.481	0.241	0.241	-0.4	1	4	0.962	.27571	3.627
234	30	3.464	2	0.727	1.213	0.83	0.479	0.174	0.291	-0.4	1.667	4	0.959	.26498	3.774

Reference Number	θ	μ_x (*σ)	μ_y (*σ)	σ_x (*σ)	σ_y (*σ)	μ_x (*CEP)	μ_y (*CEP)	σ_x (*CEP)	σ_y (*CEP)	ρ	σ_y/σ_x	bias (*σ)	bias (*CEP)	CEP_{MPI} (*CEP)	CEP/CEP_{MPI}
235	30	3.464	2	0.277	1.387	0.853	0.492	0.068	0.341	-0.4	5	4	0.984	.24007	4.165
236	30	3.464	2	1.387	0.277	0.863	0.498	0.346	0.069	0	0.2	4	0.997	.24388	4.1
237	30	3.464	2	1.213	0.728	0.849	0.49	0.297	0.178	0	0.6	4	0.98	.27768	3.601
238	30	3.464	2	1	1	0.84	0.485	0.242	0.242	0	1	4	0.97	.28574	3.5
239	30	3.464	2	0.727	1.213	0.839	0.484	0.176	0.294	0	1.667	4	0.968	.27445	3.644
240	30	3.464	2	0.277	1.387	0.855	0.494	0.068	0.342	0	5	4	0.987	.24186	4.135
241	30	3.464	2	1.387	0.277	0.864	0.499	0.346	0.069	0.4	0.2	4	0.998	.2429	4.117
242	30	3.464	2	1.213	0.728	0.855	0.493	0.299	0.18	0.4	0.6	4	0.987	.27258	3.669
243	30	3.464	2	1	1	0.849	0.49	0.245	0.245	0.4	1	4	0.981	.28108	3.558
244	30	3.464	2	0.727	1.213	0.849	0.49	0.178	0.297	0.4	1.667	4	0.981	.27107	3.689
245	30	3.464	2	0.277	1.387	0.86	0.496	0.069	0.344	0.4	5	4	0.993	.24208	4.131
246	30	3.464	2	1.387	0.277	0.865	0.5	0.346	0.069	0.8	0.2	4	0.999	.24021	4.163
247	30	3.464	2	1.213	0.728	0.862	0.498	0.302	0.181	0.8	0.6	4	0.995	.25129	3.979
248	30	3.464	2	1	1	0.86	0.497	0.248	0.248	0.8	1	4	0.993	.25615	3.904
249	30	3.464	2	0.727	1.213	0.861	0.497	0.181	0.301	0.8	1.667	4	0.994	.25119	3.981
250	30	3.464	2	0.277	1.387	0.874	0.505	0.07	0.35	0.8	5	4	0.999	.24306	4.114
251	60	2	3.464	1.387	0.277	0.493	0.853	0.342	0.068	-0.8	0.2	4	0.985	.23685	4.222
252	60	2	3.464	1.213	0.728	0.477	0.826	0.289	0.174	-0.8	0.6	4	0.954	.2409	4.151
253	60	2	3.464	1	1	0.481	0.833	0.24	0.24	-0.8	1	4	0.962	.24794	4.033
254	60	2	3.464	0.727	1.213	0.492	0.852	0.179	0.298	-0.8	1.667	4	0.983	.24856	4.023
255	60	2	3.464	0.277	1.387	0.494	0.855	0.068	0.342	-0.8	5	4	0.987	.2378	4.205
256	60	2	3.464	1.387	0.277	0.493	0.854	0.342	0.068	-0.4	0.2	4	0.986	.24013	4.164
257	60	2	3.464	1.213	0.728	0.479	0.83	0.291	0.174	-0.4	0.6	4	0.958	.26469	3.778
258	60	2	3.464	1	1	0.481	0.833	0.24	0.24	-0.4	1	4	0.962	.27566	3.628
259	60	2	3.464	0.727	1.213	0.489	0.846	0.178	0.296	-0.4	1.667	4	0.977	.27017	3.701
260	60	2	3.464	0.277	1.387	0.498	0.863	0.069	0.345	-0.4	5	4	0.996	.24291	4.117
261	60	2	3.464	1.387	0.277	0.495	0.857	0.343	0.069	0	0.2	4	0.989	.24205	4.131

Reference Number	θ	μ_x (*σ)	μ_y (*σ)	σ_x (*σ)	σ_y (*σ)	μ_x (*CEP)	μ_y (*CEP)	σ_x (*CEP)	σ_y (*CEP)	ρ	σ_y/σ_x	bias (*σ)	bias (*CEP)	CEP_{MPI} (*CEP)	CEP/CEP_{MPI}
262	60	2	3.464	1.213	0.728	0.484	0.838	0.293	0.176	0	0.6	4	0.968	.27418	3.647
263	60	2	3.464	1	1	0.485	0.84	0.242	0.242	0	1	4	0.97	.2857	3.5
264	60	2	3.464	0.727	1.213	0.49	0.849	0.178	0.297	0	1.667	4	0.98	.27786	3.599
265	60	2	3.464	0.277	1.387	0.499	0.863	0.069	0.346	0	5	4	0.997	.24429	4.094
266	60	2	3.464	1.387	0.277	0.497	0.86	0.344	0.069	0.4	0.2	4	0.994	.24193	4.133
267	60	2	3.464	1.213	0.728	0.49	0.849	0.297	0.178	0.4	0.6	4	0.98	.27081	3.693
268	60	2	3.464	1	1	0.49	0.849	0.245	0.245	0.4	1	4	0.981	.28106	3.558
269	60	2	3.464	0.727	1.213	0.493	0.855	0.179	0.299	0.4	1.667	4	0.987	.27279	3.666
270	60	2	3.464	0.277	1.387	0.499	0.865	0.069	0.346	0.4	5	4	0.998	.24344	4.108
271	60	2	3.464	1.387	0.277	0.499	0.864	0.346	0.069	0.8	0.2	4	0.998	.23988	4.169
272	60	2	3.464	1.213	0.728	0.497	0.86	0.301	0.181	0.8	0.6	4	0.994	.25087	3.986
273	60	2	3.464	1	1	0.497	0.86	0.248	0.248	0.8	1	4	0.993	.25613	3.904
274	60	2	3.464	0.727	1.213	0.498	0.862	0.181	0.302	0.8	1.667	4	0.995	.25157	3.975
275	60	2	3.464	0.277	1.387	0.5	0.866	0.069	0.347	0.8	5	4	0.999	.24078	4.153

Appendix D: The Sample Analysis Sets For Each Sample Size

In this appendix, the reference number for each sample analysis set for a given sample size is listed.

NOTATION KEY

s_x = sample crossrange standard deviation s_y = sample downrange standard deviation

$\bar{\sigma} = \sqrt{\frac{S_x^2 + S_y^2}{2}}$ $\bar{\rho}$ = sample correlation

Sample Analysis Set Reference Number	Bias Range	$\bar{\rho}$ Range	s_y/s_x Range
1	$[0, 0.75\bar{\sigma}]$	(-0.6, -1.0)	(< 0.4)
2	$[0, 0.75\bar{\sigma}]$	(-0.6, -1.0)	[0.4, 0.8)
3	$[0, 0.75\bar{\sigma}]$	(-0.6, -1.0)	[0.8, 1.25]
4	$[0, 0.75\bar{\sigma}]$	(-0.6, -1.0)	(1.25, 2.5]
5	$[0, 0.75\bar{\sigma}]$	(-0.6, -1.0)	(> 2.5)
6	$[0, 0.75\bar{\sigma}]$	(-0.2, -0.6]	(< 0.4)
7	$[0, 0.75\bar{\sigma}]$	(-0.2, -0.6]	[0.4, 0.8)
8	$[0, 0.75\bar{\sigma}]$	(-0.2, -0.6]	[0.8, 1.25]
9	$[0, 0.75\bar{\sigma}]$	(-0.2, -0.6]	(1.25, 2.5]
10	$[0, 0.75\bar{\sigma}]$	(-0.2, -0.6]	(> 2.5)
11	$[0, 0.75\bar{\sigma}]$	[-0.2, 0.2]	(< 0.4)
12	$[0, 0.75\bar{\sigma}]$	[-0.2, 0.2]	[0.4, 0.8)
13	$[0, 0.75\bar{\sigma}]$	[-0.2, 0.2]	[0.8, 1.25]
14	$[0, 0.75\bar{\sigma}]$	[-0.2, 0.2]	(1.25, 2.5]
15	$[0, 0.75\bar{\sigma}]$	[-0.2, 0.2]	(> 2.5)
16	$[0, 0.75\bar{\sigma}]$	(0.2, 0.6]	(< 0.4)
17	$[0, 0.75\bar{\sigma}]$	(0.2, 0.6]	[0.4, 0.8)
18	$[0, 0.75\bar{\sigma}]$	(0.2, 0.6]	[0.8, 1.25]
19	$[0, 0.75\bar{\sigma}]$	(0.2, 0.6]	(1.25, 2.5]
20	$[0, 0.75\bar{\sigma}]$	(0.2, 0.6]	(> 2.5)
21	$[0, 0.75\bar{\sigma}]$	(0.6, 1.0]	(< 0.4)
22	$[0, 0.75\bar{\sigma}]$	(0.6, 1.0]	[0.4, 0.8)
23	$[0, 0.75\bar{\sigma}]$	(0.6, 1.0]	[0.8, 1.25]
24	$[0, 0.75\bar{\sigma}]$	(0.6, 1.0]	(1.25, 2.5]
25	$[0, 0.75\bar{\sigma}]$	(0.6, 1.0]	(> 2.5)
26	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	(-0.6, -1.0)	(< 0.4)
27	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	(-0.6, -1.0)	[0.4, 0.8)
28	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	(-0.6, -1.0)	[0.8, 1.25]

Sample Analysis Set Reference Number	Bias Range	$\bar{\rho}$ Range	s_y/s_x Range
29	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	(-0.6, -1.0)	(1.25, 2.5]
30	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	(-0.6, -1.0)	(> 2.5)
31	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	(-0.2, -0.6]	(< 0.4)
32	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	(-0.2, -0.6]	[0.4, 0.8)
33	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	(-0.2, -0.6]	[0.8, 1.25]
34	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	(-0.2, -0.6]	(1.25, 2.5]
35	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	(-0.2, -0.6]	(> 2.5)
36	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	[-0.2, 0.2]	(< 0.4)
37	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	[-0.2, 0.2]	[0.4, 0.8)
38	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	[-0.2, 0.2]	[0.8, 1.25]
39	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	[-0.2, 0.2]	(1.25, 2.5]
40	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	[-0.2, 0.2]	(> 2.5)
41	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	(0.2, 0.6]	(< 0.4)
42	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	(0.2, 0.6]	[0.4, 0.8)
43	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	(0.2, 0.6]	[0.8, 1.25]
44	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	(0.2, 0.6]	(1.25, 2.5]
45	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	(0.2, 0.6]	(> 2.5)
46	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	(0.6, 1.0]	(< 0.4)
47	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	(0.6, 1.0]	[0.4, 0.8)
48	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	(0.6, 1.0]	[0.8, 1.25]
49	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	(0.6, 1.0]	(1.25, 2.5]
50	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	(0.6, 1.0]	(> 2.5)
51	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	(-0.6, -1.0)	(< 0.4)
52	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	(-0.6, -1.0)	[0.4, 0.8)
53	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	(-0.6, -1.0)	[0.8, 1.25]
54	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	(-0.6, -1.0)	(1.25, 2.5]
55	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	(-0.6, -1.0)	(> 2.5)
56	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	(-0.2, -0.6]	(< 0.4)
57	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	(-0.2, -0.6]	[0.4, 0.8)
58	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	(-0.2, -0.6]	[0.8, 1.25]
59	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	(-0.2, -0.6]	(1.25, 2.5]
60	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	(-0.2, -0.6]	(> 2.5)
61	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	[-0.2, 0.2]	(< 0.4)
62	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	[-0.2, 0.2]	[0.4, 0.8)
63	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	[-0.2, 0.2]	[0.8, 1.25]
64	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	[-0.2, 0.2]	(1.25, 2.5]
65	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	[-0.2, 0.2]	(> 2.5)
66	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	(0.2, 0.6]	(< 0.4)
67	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	(0.2, 0.6]	[0.4, 0.8)

Sample Analysis Set Reference Number	Bias Range	$\bar{\rho}$ Range	s_y / s_x Range
68	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	(0.2, 0.6]	[0.8, 1.25]
69	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	(0.2, 0.6]	(1.25, 2.5]
70	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	(0.2, 0.6]	(> 2.5)
71	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	(0.6, 1.0]	(< 0.4)
72	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	(0.6, 1.0]	[0.4, 0.8)
73	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	(0.6, 1.0]	[0.8, 1.25]
74	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	(0.6, 1.0]	(1.25, 2.5]
75	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	(0.6, 1.0]	(> 2.5)
76	$(> 2.75\bar{\sigma})$	(-0.6, -1.0)	(< 0.4)
77	$(> 2.75\bar{\sigma})$	(-0.6, -1.0)	[0.4, 0.8)
78	$(> 2.75\bar{\sigma})$	(-0.6, -1.0)	[0.8, 1.25]
79	$(> 2.75\bar{\sigma})$	(-0.6, -1.0)	(1.25, 2.5]
80	$(> 2.75\bar{\sigma})$	(-0.6, -1.0)	(> 2.5)
81	$(> 2.75\bar{\sigma})$	(-0.2, -0.6]	(< 0.4)
82	$(> 2.75\bar{\sigma})$	(-0.2, -0.6]	[0.4, 0.8)
83	$(> 2.75\bar{\sigma})$	(-0.2, -0.6]	[0.8, 1.25]
84	$(> 2.75\bar{\sigma})$	(-0.2, -0.6]	(1.25, 2.5]
85	$(> 2.75\bar{\sigma})$	(-0.2, -0.6]	(> 2.5)
86	$(> 2.75\bar{\sigma})$	[-0.2, 0.2]	(< 0.4)
87	$(> 2.75\bar{\sigma})$	[-0.2, 0.2]	[0.4, 0.8)
88	$(> 2.75\bar{\sigma})$	[-0.2, 0.2]	[0.8, 1.25]
89	$(> 2.75\bar{\sigma})$	[-0.2, 0.2]	(1.25, 2.5]
90	$(> 2.75\bar{\sigma})$	[-0.2, 0.2]	(> 2.5)
91	$(> 2.75\bar{\sigma})$	(0.2, 0.6]	(< 0.4)
92	$(> 2.75\bar{\sigma})$	(0.2, 0.6]	[0.4, 0.8)
93	$(> 2.75\bar{\sigma})$	(0.2, 0.6]	[0.8, 1.25]
94	$(> 2.75\bar{\sigma})$	(0.2, 0.6]	(1.25, 2.5]
95	$(> 2.75\bar{\sigma})$	(0.2, 0.6]	(> 2.5)
96	$(> 2.75\bar{\sigma})$	(0.6, 1.0]	(< 0.4)
97	$(> 2.75\bar{\sigma})$	(0.6, 1.0]	[0.4, 0.8)
98	$(> 2.75\bar{\sigma})$	(0.6, 1.0]	[0.8, 1.25]
99	$(> 2.75\bar{\sigma})$	(0.6, 1.0]	(1.25, 2.5]
100	$(> 2.75\bar{\sigma})$	(0.6, 1.0]	(> 2.5)

Appendix E: The MODSIM Sample Generator Program

For an input sample size n and μ_x, μ_y, σ_x, σ_y, and ρ values from an input data file, this program was used to create n simulated (x, y) coordinates. The program generates sample statistics and other data required for input into the MathCAD CEP estimator template. The program as displayed was used to generate the sample output found in Table 3.4 on page 31.

```
MAIN MODULE samp ;

FROM IOMod IMPORT StreamObj, ALL FileUseType ;
FROM RandMod IMPORT RandomObj ;
FROM MathMod IMPORT EXP, LN, SIN, COS, POWER,ATAN, pi;

VAR
        sin, sout : StreamObj ;

        ux, uy, ox, oy, p, x, y, theta, xsum, ysum, rbar, biasum, ot2,
        uxtrans, uytrans, oxtrans, oytrans, xtrans, ytrans, tsum, tbar,
        xbar, ybar, sx, sy, pbar, nreal, sxsum, sysum, psum, b, stsum,
        knum, kden, ks, med, sd, smedian, u, d, t, dsum, w, ethridge
        : REAL ;

        n, k, h, i, j, q, rep, input, stop : INTEGER;

        median : ARRAY INTEGER OF REAL ;

        space : STRING ;

{ In MODSIM, subroutines are called "procedures" and must
  be written PRIOR to the main body}

PROCEDURE compute(IN ux,uy,ox,oy,p:REAL;
                        OUT a,uxt,uyt,oxt,oyt:REAL) ;
      VAR
        square1, square2 : REAL ;

      BEGIN  { This procedure finds the transformed values of
            ox, oy, ux, and uy }
        IF ox <> oy
          a := 0.5*(ATAN((2.0*p*ox*oy)/(ox*ox-oy*oy))) ;
        ELSIF p > 0.0
          a := -1.0*pi/4.0 ;
```

```
        ELSIF p < 0.0
          a := pi/4.0 ;
        ELSIF p = 0.0
          a := 0.0 ;
        END IF ;
        uxt := ux*COS(a) + uy*SIN(a) ;
        uyt := -1.0*ux*SIN(a) + uy*COS(a) ;
        square1:=(ox*ox-oy*oy)*(ox*ox-oy*oy)+4.0*p*p*ox*ox*oy*oy;
        square2 := POWER(square1, 0.5);
        IF ox > oy
          oxt := POWER(((ox*ox + oy*oy + square2)/2.0),0.5);
          oyt := POWER(((ox*ox + oy*oy - square2)/2.0),0.5);
        ELSE
          oyt := POWER(((ox*ox + oy*oy + square2)/2.0),0.5);
          oxt := POWER(((ox*ox + oy*oy - square2)/2.0),0.5);
        END IF ;
END PROCEDURE ;

PROCEDURE select (IN uxt,uyt,oxt,oyt:REAL; IN k:INTEGER;
                    OUT xtrans,ytrans:REAL) ;
      VAR
        r1,r2 : RandomObj ;
        k2 : INTEGER ;

      BEGIN          {This procedure selects a random x and y value from }
        NEW(r1) ;    {the bivariate normal distribution}
        NEW(r2) ;
        k2 := k*10 ;
        ASK r1 TO SetSeed(k) ;
        ASK r2 TO SetSeed(k2) ;
        xtrans := ASK r1 TO Normal (uxt, oxt) ;
        ytrans := ASK r2 TO Normal (uyt, oyt) ;
        DISPOSE(r1) ;
        DISPOSE(r2) ;
END PROCEDURE ;

BEGIN  {main body}

input := 35 ;   { You must manually enter the number of data sets in
                  the input file name in the following statement }

n := 1000 ;     { You must manually enter the desired sample size n for
                  each sample set created in the following statement }
```

```
OUTPUT(" n   pbar    xbar    ybar    sx      sy    ",
          "smedian   rbar   ethridge");
OUTPUT ;
nreal := FLOAT(n) ;
k := 1 ;                { k = seed position, must be manually entered.}
xsum := 0.0 ;
ysum := 0.0 ;           { For n = 3, set k = 1 }
sxsum := 0.0 ;          { For n = 6, set k = 50000 }
sysum := 0.0 ;          { For n = 9, set k = 100000 }
psum := 0.0 ;           { For n = 15, set k = 150000 }
sd := 0.0 ;
biasum := 0.0 ;
knum := 0.0 ;
stsum := 0.0 ;
dsum := 0.0 ;
u := 0.0 ;
tsum := 0.0 ;
smedian := 0.0 ;
NEW(sin) ;
space := "     " ;

{ You must manually enter the input file name into the
         following statement }

ASK sin TO Open ("d.txt",Input) ;

{ The input file name is entered inside the italices (" ") }

FOR h := 1 TO input
        ASK sin TO ReadInt(rep) ;
        ASK sin TO ReadReal(ux) ;
        ASK sin TO ReadReal(uy) ;
        ASK sin TO ReadReal(ox) ;
        ASK sin TO ReadReal(oy) ;
        ASK sin TO ReadReal(p) ;
        FOR i := 1 TO rep
          NEW (median, 1..n) ;
          compute(ux,uy,ox,oy,p,theta,uxtrans,uytrans,oxtrans,oytrans);
          OUTPUT("Analysis sample #",h,"    rep # ",i," :") ;
          OUTPUT ;
          FOR j := 1 TO n
            k := k+1 ;
              select(uxtrans,uytrans,oxtrans,oytrans,k,xtrans,ytrans) ;
              x := xtrans*COS(theta)-ytrans*SIN(theta) ;
```

```
    y := xtrans*SIN(theta)+ytrans*COS(theta) ;
  xsum := xsum + x ;
  ysum := ysum + y ;
  b := POWER((x*x+y*y),0.5);
  median[j] := b ;
  biasum := biasum + b ;
  tsum := tsum + LN(b) ;
 { OUTPUT("x = ",x, space,"y = ",y) ; }
END FOR ;
xbar := xsum/nreal ;
ybar := ysum/nreal ;
tbar := tsum/nreal ;
rbar := biasum/nreal ;
k := k-n ;

{ routine for finding the sample median }

IF ODD(n) = TRUE
    stop := TRUNC(nreal/2.0) ;
  FOR j := 1 TO stop
    med := 0.0 ;
      FOR q := 1 TO n
      med := MAXOF(median[q],med);
      END FOR ;
    FOR q := 1 TO n
      IF median[q] = med
        median [q] := 0.0 ;
      END IF ;
    END FOR ;
    j := j + 1 ;
  END FOR ;
  smedian := 0.0 ;
  FOR q := 1 TO n
    smedian := MAXOF(median[q],smedian) ;
  END FOR ;
ELSE
  stop := TRUNC(nreal/2.0) ;
  FOR j := 1 TO stop
    med := 0.0 ;
      FOR q := 1 TO n
      med := MAXOF(median[q],med);
      END FOR ;
    FOR q := 1 TO n
      IF median[q] = med
```

```
          median [q] := 0.0 ;
        END IF ;
      END FOR ;
    j := j + 1 ;
    END FOR ;
    smedian := 0.0 ;
    FOR q := 1 TO n
      smedian := MAXOF(median[q],smedian) ;
    END FOR ;
    smedian := (smedian + med)/2.0 ;
  END IF ;

  FOR j := 1 TO n
    k := k+1 ;
select(uxtrans,uytrans,oxtrans,oytrans,k,xtrans,ytrans) ;
    x := xtrans*COS(theta)-ytrans*SIN(theta) ;
          y := xtrans*SIN(theta)+ytrans*COS(theta) ;
          b := POWER((x*x+y*y),0.5);
    t := LN(b) ;
    sxsum := sxsum + POWER((x - xbar),2.0) ;
    sysum := sysum + POWER((y - ybar),2.0) ;
    psum := psum + (x-xbar)*(y-ybar) ;
    knum := knum + POWER(t-tbar,4.0);
    stsum := stsum + POWER(t-tbar,2.0);
  END FOR ;
  sx := POWER((sxsum/(nreal-1.0)),0.5) ;
  sy := POWER((sysum/(nreal-1.0)),0.5) ;
  pbar := psum/((nreal-1.0)*sx*sy) ;
  ot2 := stsum/(nreal - 1.0) ;
  kden := POWER(stsum, 2.0) ;
  ks := knum/kden ;
  k := k-n ;
  FOR j := 1 TO n
    k := k+1 ;
      select(uxtrans,uytrans,oxtrans,oytrans,k,xtrans,ytrans) ;
    x := xtrans*COS(theta)-ytrans*SIN(theta) ;
    y := xtrans*SIN(theta)+ytrans*COS(theta) ;
    b := POWER((x*x+y*y),0.5);
    t := LN(b) ;
    d := 1.0 -
      (0.03*(POWER((ks-3.0),3.0))*(t-smedian)*(t-smedian))/ot2;
    IF d <= 0.01
      d := 0.01 ;
    END IF ;
```

```
      dsum := dsum + (1.0/d) ;
    END FOR ;
    k := k-n ;

    FOR j := 1 TO n
      k := k+1 ;
      select(uxtrans,uytrans,oxtrans,oytrans,k,xtrans,ytrans) ;
        x := xtrans*COS(theta)-ytrans*SIN(theta) ;
      y := xtrans*SIN(theta)+ytrans*COS(theta) ;
      b := POWER((x*x+y*y),0.5);
      t := LN(b) ;
      d := 1.0-
       (0.03*(POWER((ks-3.0),3.0))*(t-smedian)*(t-smedian))/ot2;
      IF d <= 0.01
        d := 0.01 ;
      END IF ;
      w := (1.0/d) / dsum ;
      u := u + w*t ;
    END FOR ;
    ethridge := EXP(u) ;

    OUTPUT(n," ",pbar," ",xbar," ",ybar," ",sx," ",sy," ",
         smedian," ",rbar," ",ethridge);

    xsum := 0.0 ;
    ysum := 0.0 ;
    sxsum := 0.0 ;
    sysum := 0.0 ;
    psum := 0.0 ;
    DISPOSE (median) ;
    sd := 0.0 ;
    biasum := 0.0 ;
    knum := 0.0 ;
    stsum := 0.0 ;
    dsum := 0.0 ;
    u := 0.0 ;
    smedian := 0.0 ;
    tsum := 0.0 ;
  END FOR ;
END FOR ;
ASK sin TO Close ;
DISPOSE (sin) ;

END MODULE.
```

Appendix F: The MathCAD CEP Estimator Template

This template takes in output from the MODSIM sample generator program. It outputs the CEP estimate for the each of the 8 thesis CEP estimators for each input simulation run. In the example displayed, the input is from design point 20 for sample size 15:

ORIGIN := 1

TOL := 0.01

input := READPRN (pop15)

a := 191

b := 200

n := 10

s := submatrix (input, a, b, 1, 9)

i := 1.. n

Input Variable Identification:

$$
\begin{array}{c}
\begin{array}{ccccccccc} n & \text{pbar} & \text{xbar} & \text{ybar} & s_x & s_y & \text{Smed} & \text{rbar} & \text{Ethridge} \end{array} \\
s = \begin{bmatrix}
15 & 0.141 & 47.57 & 8.641 & 21.065 & 87.89 & 93.568 & 91.499 & 82.542 \\
15 & 0.085 & 44.637 & 26.311 & 18.122 & 151.461 & 84.27 & 130.741 & 108.669 \\
15 & 0.044 & 47.788 & 56.131 & 18.011 & 123.134 & 109.932 & 124.324 & 104.501 \\
15 & -0.429 & 41.111 & 11.558 & 31.451 & 118.894 & 119.437 & 117.336 & 105.475 \\
15 & -0.026 & 44.979 & 25.827 & 24.911 & 109.761 & 83.48 & 102.614 & 88.238 \\
15 & 0.037 & 37.786 & 35.591 & 21.951 & 148.713 & 114.44 & 133.202 & 113.387 \\
15 & 0.57 & 45.209 & 19.972 & 24.782 & 120.626 & 108.014 & 113.531 & 95.339 \\
15 & -0.029 & 50.522 & 23.133 & 28.195 & 113.19 & 107.833 & 116.402 & 105.595 \\
15 & -0.158 & 35.351 & 16.15 & 29.818 & 106.429 & 89.328 & 97.421 & 79.691 \\
15 & -0.49 & 44.933 & 35.422 & 27.796 & 172.219 & 125.115 & 152.514 & 126.34
\end{bmatrix}
\end{array}
$$

Smed: Sample Median Estimator

$$CEP_{i,1} := s_{i,7} \qquad re1_{i,1} := \frac{\left|CEP_{i,1} - 100\right|}{100} \qquad sre1_{i,1} := \left(\frac{\left|CEP_{i,1} - 100\right|}{100}\right)^2$$

Ethridge: Ethridge CEP Estimator

$$CEP_{i,2} := s_{i,9} \qquad re2_{i,1} := \frac{\left|CEP_{i,2} - 100\right|}{100} \qquad sre2_{i,1} := \left(\frac{\left|CEP_{i,2} - 100\right|}{100}\right)^2$$

MRand: Modified Rand-234 CEP Estimator

$$\sigma s_i := \sqrt{\frac{\left(s_{i,5}\right)^2 + \left(s_{i,6}\right)^2 - \sqrt{\left[\left(s_{i,5}\right)^2 - \left(s_{i,6}\right)^2\right]^2 + 4\cdot\left(s_{i,2}\right)^2\cdot\left(s_{i,5}\right)^2\cdot\left(s_{i,6}\right)^2}}{2}}$$

$$\sigma L_i := \sqrt{\frac{\left(s_{i,5}\right)^2 + \left(s_{i,6}\right)^2 + \sqrt{\left[\left(s_{i,5}\right)^2 - \left(s_{i,6}\right)^2\right]^2 + 4\cdot\left(s_{i,2}\right)^2\cdot\left(s_{i,5}\right)^2\cdot\left(s_{i,6}\right)^2}}{2}}$$

$$CEPMPI_i := .563\cdot\sigma L_i + .614\cdot\sigma s_i \qquad v_i := \frac{\sqrt{\left(s_{i,3}\right)^2 + \left(s_{i,4}\right)^2}}{CEPMPI_i} \qquad c_i := \frac{\sigma s_i}{\sigma L_i}$$

This estimator is not reliable if either $v > 2.2$ or $c \le .25$

$$CEP_{i,3} := \left[CEPMPI_i\cdot\left[1.0039 - .0528\cdot v_i + .4786\cdot\left(v_i\right)^2 - .0793\cdot\left(v_i\right)^3\right]\right]$$

$$re3_{i,1} := \frac{\left|CEP_{i,3} - 100\right|}{100} \qquad sre3_{i,1} := \left(\frac{\left|CEP_{i,3} - 100\right|}{100}\right)^2$$

Valstar: Valstar CEP Estimator

$$\sigma s_i := \sqrt{\frac{\left(s_{i,5}\right)^2 + \left(s_{i,6}\right)^2 - \sqrt{\left[\left(s_{i,5}\right)^2 - \left(s_{i,6}\right)^2\right]^2 + 4\cdot\left(s_{i,2}\right)^2\cdot\left(s_{i,5}\right)^2\cdot\left(s_{i,6}\right)^2}}{2}}$$

$$\sigma L_i := \sqrt{\frac{\left(s_{i,5}\right)^2 + \left(s_{i,6}\right)^2 + \sqrt{\left[\left(s_{i,5}\right)^2 - \left(s_{i,6}\right)^2\right]^2 + 4\cdot\left(s_{i,2}\right)^2\cdot\left(s_{i,5}\right)^2\cdot\left(s_{i,6}\right)^2}}{2}} \qquad c_i := \frac{\sigma s_i}{\sigma L_i}$$

$$CEPMPI_i := \left| \begin{array}{ll} .562\cdot\sigma L_i + .615\cdot\sigma s_i & \text{if } .369 < c_i \le 1 \\ .675\cdot\sigma L_i + \dfrac{\sigma s_i}{1.2\cdot\sigma L_i} & \text{if } 0 \le c_i \le .369 \end{array} \right.$$

$$CEP_{i,4} := \sqrt{\left(CEPMPI_i\right)^2 + \left(s_{i,3}\right)^2 + \left(s_{i,4}\right)^2}$$

$$re4_{i,1} := \frac{\left|CEP_{i,4} - 100\right|}{100} \qquad sre4_{i,1} := \left(\frac{\left|CEP_{i,4} - 100\right|}{100}\right)^2$$

Grubbs: Grubbs-Patniak Chi-Square CEP Estimator

$$m_i := \left(s_{i,3}\right)^2 + \left(s_{i,4}\right)^2 + \left(s_{i,5}\right)^2 + \left(s_{i,6}\right)^2$$

$$v2_i := \left[\left(s_{i,3}\right)^2 \cdot \left(s_{i,5}\right)^2 + 2 \cdot s_{i,3} \cdot s_{i,4} \cdot s_{i,2} \cdot s_{i,5} \cdot s_{i,6} + \left(s_{i,4}\right)^2 \cdot \left(s_{i,6}\right)^2\right]$$

$$v_i := 2 \cdot \left[\left(s_{i,5}\right)^4 + 2 \cdot \left(s_{i,2}\right)^2 \cdot \left(s_{i,5}\right)^2 \cdot \left(s_{i,6}\right)^2 + \left(s_{i,6}\right)^4\right] + 4 \cdot v2_i \qquad d_i := \frac{2 \cdot \left(m_i\right)^2}{v_i}$$

$$CEP_{i,5} := \sqrt{\frac{v_i \cdot \mathrm{qchisq}\left(.5, d_i\right)}{2 \cdot m_i}} \qquad re5_{i,1} := \frac{\left|CEP_{i,5} - 100\right|}{100} \qquad sre5_{i,1} := \left(\frac{\left|CEP_{i,5} - 100\right|}{100}\right)^2$$

Rayleigh:
Rayleigh Distribution
CEP Estimator

$$CEP_{i,6} := .9394 \cdot s_{i,8} \qquad re6_{i,1} := \frac{\left|CEP_{i,6} - 100\right|}{100} \qquad sre6_{i,1} := \left(\frac{\left|CEP_{i,6} - 100\right|}{100}\right)^2$$

Numerical:
The Direct Numerical Integration CEP Estimator:

$$\Omega(x,y) := \frac{1}{\left[2 \cdot \left[1 - \left(s_{i,2}\right)^2\right]\right]} \cdot \left[\left(\frac{x - s_{i,3}}{s_{i,5}}\right)^2 - 2 \cdot s_{i,2} \cdot \left[\frac{\left(x - s_{i,3}\right) \cdot \left(y - s_{i,4}\right)}{s_{i,5} \cdot s_{i,6}}\right] + \left(\frac{y - s_{i,4}}{s_{i,6}}\right)^2\right]$$

$$f(x,y) := \frac{1}{\left[2 \cdot \pi \cdot s_{i,5} \cdot s_{i,6} \cdot \sqrt{1 - \left(s_{i,2}\right)^2}\right]} \cdot e^{-\Omega(x,y)}$$ = the joint bivariate normal distribution.

$$r_i := CEP_{i,5}$$

$$pr(i,r) := \mathrm{root}\left[\int_{-r}^{r} \int_{-\sqrt{r^2 - y^2}}^{\sqrt{r^2 - y^2}} f(x,y)\, dx\, dy - .5, r\right]$$

$$CEP_{i,7} := pr\left(i, r_i\right) \qquad re7_{i,1} := \frac{\left|CEP_{i,7} - 100\right|}{100} \qquad sre7_{i,1} := \left(\frac{\left|CEP_{i,7} - 100\right|}{100}\right)^2$$

TMCBN: Tongue's Modified CBN (Correlated Bivariate Normal) Estimator

$$CBN_i := CEP_{i,7} \qquad STDR_i := \begin{cases} \dfrac{s_{i,5}}{s_{i,6}} & \text{if } s_{i,5} \le s_{i,6} \\[2ex] \dfrac{s_{i,6}}{s_{i,5}} & \text{if } s_{i,5} > s_{i,6} \end{cases} \qquad CORR_i := s_{i,2} \qquad BIAS_i := \frac{\sqrt{\left(s_{i,3}\right)^2 + \left(s_{i,4}\right)^2}}{\sqrt{\dfrac{\left(s_{i,5}\right)^2 + \left(s_{i,6}\right)^2}{2}}}$$

$$RE1_i := .171833 - .009784 \cdot s_{i,1} - .037707 \cdot BIAS_i - .150628 \cdot STDR_i + .002045 \cdot s_{i,1} \cdot BIAS_i + .007488 \cdot s_{i,1} \cdot STDR_i$$

$$RE2_i := .019014 \cdot BIAS_i \cdot STDR_i + .116385 \cdot STDR_i \cdot CORR_i - .006714 \cdot STDR_i \cdot CORR_i \cdot s_{i,1}$$

$$RE_i := RE1_i + RE2_i$$

$$CEP_{i,8} := CBN_i \cdot \left(1 - RE_i\right) \qquad re8_{i,1} := \frac{\left|CEP_{i,8} - 100\right|}{100} \qquad sre8_{i,1} := \left(\frac{\left|CEP_{i,8} - 100\right|}{100}\right)^2$$

---

mre1 := mean(re1)	mre2 := mean(re2)	mre3 := mean(re3)	mre4 := mean(re4)
mre5 := mean(re5)	mre6 := mean(re6)	mre7 := mean(re7)	mre8 := mean(re8)
vre1 := var(re1)	vre2 := var(re2)	vre3 := var(re3)	vre4 := var(re4)
vre5 := var(re5)	vre6 := var(re6)	vre7 := var(re7)	vre8 := var(re8)
msre1 := mean(sre1)	msre2 := mean(sre2)	msre3 := mean(sre3)	msre4 := mean(sre4)
msre5 := mean(sre5)	msre6 := mean(sre6)	msre7 := mean(sre7)	msre8 := mean(sre8)

mre1 = 0.134	mre2 = 0.118	mre3 = 0.111	mre4 = 0.138
vre1 = 0.0032	vre2 = 0.0051	vre3 = 0.0046	vre4 = 0.0059
msre1 = 0.021	msre2 = 0.019	msre3 = 0.017	msre4 = 0.025
mre5 = 0.109	mre6 = 0.16	mre7 = 0.104	mre8 = 0.103
vre5 = 0.009	vre6 = 0.0125	vre7 = 0.0077	vre8 = 0.0065
msre5 = 0.021	msre6 = 0.038	msre7 = 0.019	msre8 = 0.017

mre := (mre1 mre2 mre3 mre4 mre5 mre6 mre7 mre8) = the mean relative error values

vre := (vre1 vre2 vre3 vre4 vre5 vre6 vre7 vre8) = the variance from the MRE values

msre := (msre1 msre2 msre3 msre4 msre5 msre6 msre7 msre8) = the MSRE values

Output Identification

$$
\begin{array}{c c}
 & \begin{array}{cccccccc} \text{Smed} & \text{Ethridge} & \text{MRand} & \text{Valstar} & \text{Grubbs} & \text{Rayleigh} & \text{Numerical} & \text{TMCBN} \end{array} \\
\text{CEP} = & \begin{bmatrix}
93.568 & 82.542 & 75.646 & 76.713 & 82.771 & 85.955 & 81.902 & 80.71 \\
84.27 & 108.669 & 106.15 & 114.71 & 116.91 & 122.818 & 114.744 & 112.637 \\
109.932 & 104.501 & 104.237 & 111.19 & 108.723 & 116.79 & 105.845 & 104.15 \\
119.437 & 105.475 & 92.236 & 91.564 & 98.623 & 110.226 & 98.134 & 96.939 \\
83.48 & 88.238 & 89.489 & 90.595 & 95.334 & 96.396 & 94.431 & 93.061 \\
114.44 & 113.387 & 106.928 & 113.119 & 114.252 & 125.13 & 112.713 & 110.747 \\
108.014 & 95.339 & 91.511 & 95.858 & 98.96 & 106.651 & 97.213 & 95.535 \\
107.833 & 105.595 & 94.579 & 94.643 & 101.094 & 109.348 & 100.49 & 99.096 \\
89.328 & 79.691 & 84.794 & 81.948 & 88.988 & 91.517 & 89.043 & 87.903 \\
125.115 & 126.34 & 122.308 & 130.003 & 132.812 & 143.272 & 130.782 & 128.714
\end{bmatrix}
\end{array}
$$

$$
\begin{array}{l c}
 & \begin{array}{cccccccc} \text{Smed} & \text{Ethridge} & \text{MRand} & \text{Valstar} & \text{Grubbs} & \text{Rayleigh} & \text{Numerical} & \text{TMCBN} \end{array} \\
\text{msre} = & (\, 0.0212 \;\; 0.0191 \;\; 0.017 \;\; 0.0249 \;\; 0.0209 \;\; 0.0382 \;\; 0.0185 \;\; 0.0171 \,) = \text{the MSRE values} \\
\text{mre} = & (\, 0.1341 \;\; 0.1182 \;\; 0.1114 \;\; 0.1377 \;\; 0.1091 \;\; 0.1604 \;\; 0.1039 \;\; 0.103 \,) = \text{the MRE values} \\
\text{vre} = & (\, 0.0032 \;\; 0.0051 \;\; 0.0046 \;\; 0.0059 \;\; 0.009 \;\; 0.0125 \;\; 0.0077 \;\; 0.0065 \,) = \text{the VRE values}
\end{array}
$$

Appendix G: The Design Point MSRE Results

Sample Size 15

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
1	0.0757	0.0446	0.0229	0.0341	0.0313	0.0603	0.03	0.0285
2	0.0222	0.0123	0.0118	0.0125	0.012	0.0127	0.012	0.0119
3	0.0175	0.0246	0.0071	0.0071	0.0071	0.0083	0.0072	0.0072
4	0.0322	0.0151	0.0094	0.0099	0.0097	0.0062	0.0099	0.0099
5	0.0754	0.0156	0.0138	0.0173	0.0191	0.0417	0.0187	0.0177
6	0.0541	0.0233	0.0162	0.0251	0.0219	0.0395	0.0215	0.0205
7	0.0275	0.0233	0.011	0.0063	0.0106	0.0126	0.0102	0.0105
8	0.0395	0.0263	0.0146	0.0149	0.0151	0.0125	0.0154	0.0147
9	0.0551	0.0372	0.0215	0.0208	0.0212	0.0247	0.021	0.0217
10	0.063	0.0218	0.0195	0.0292	0.0226	0.0332	0.0226	0.0229
11	0.0936	0.037	0.0153	0.0265	0.0262	0.059	0.0261	0.0245
12	0.034	0.041	0.0114	0.013	0.0144	0.027	0.0135	0.0132
13	0.0218	0.0298	0.0173	0.0179	0.0187	0.0262	0.0183	0.0176
14	0.0297	0.0143	0.0133	0.0189	0.0122	0.0213	0.0112	0.011
15	0.0246	0.0218	0.0127	0.0121	0.0122	0.0172	0.0123	0.0121
16	0.0422	0.0421	0.0203	0.0286	0.0188	0.0278	0.0186	0.019
17	0.0351	0.0192	0.0169	0.0205	0.0154	0.0163	0.0155	0.0148
18	0.0143	0.0242	0.0186	0.0241	0.0197	0.0148	0.0201	0.018
19	0.0223	0.0262	0.0164	0.0154	0.015	0.0138	0.015	0.0151
20	0.0212	0.0191	0.017	0.0249	0.0209	0.0382	0.0185	0.0177
21	0.0406	0.0289	0.0273	0.0472	0.027	0.057	0.0244	0.0221
22	0.0383	0.0251	0.0265	0.0331	0.0208	0.0291	0.0197	0.0188
23	0.0138	0.0341	0.0096	0.009	0.0089	0.014	0.0087	0.0095
24	0.0124	0.0067	0.0124	0.015	0.0114	0.0077	0.0117	0.0111
25	0.0223	0.0083	0.0062	0.0078	0.0093	0.0172	0.0076	0.0067
26	0.0855	0.0317	0.0197	0.0378	0.0294	0.0502	0.0292	0.0282
27	0.0723	0.0384	0.0121	0.0176	0.0127	0.0273	0.012	0.0115
28	0.0294	0.0125	0.0204	0.0326	0.0145	0.0162	0.0127	0.0125
29	0.0265	0.0112	0.0117	0.0151	0.0098	0.0147	0.0089	0.0087
30	0.0422	0.0129	0.0112	0.0207	0.0227	0.0405	0.0187	0.0168
31	0.0395	0.0138	0.0089	0.0233	0.0172	0.0504	0.0152	0.0134
32	0.0563	0.0333	0.0194	0.0185	0.0242	0.0379	0.0233	0.0223
33	0.0286	0.0172	0.0063	0.0061	0.0101	0.0158	0.0106	0.0094
34	0.0355	0.0207	0.01	0.0145	0.0106	0.0213	0.0097	0.009
35	0.0232	0.0167	0.0125	0.0168	0.0155	0.034	0.0132	0.0122
36	0.0643	0.0197	0.0108	0.0171	0.0173	0.043	0.0158	0.0145

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
37	0.0427	0.0144	0.016	0.025	0.0187	0.0171	0.0199	0.0182
38	0.0122	0.0066	0.0072	0.0085	0.0074	0.0052	0.0077	0.007
39	0.0337	0.0194	0.0188	0.0215	0.0182	0.0186	0.0186	0.0179
40	0.0407	0.0138	0.0169	0.0218	0.0183	0.0351	0.0157	0.0141
41	0.023	0.0178	0.0099	0.0177	0.0167	0.0357	0.0155	0.0145
42	0.0713	0.0323	0.0186	0.0222	0.0178	0.028	0.0183	0.017
43	0.0193	0.0267	0.0184	0.0177	0.0193	0.0183	0.0193	0.0197
44	0.024	0.0214	0.0171	0.0236	0.0196	0.02	0.0206	0.0194
45	0.0222	0.0093	0.0138	0.0159	0.0103	0.0153	0.0092	0.0091
46	0.0707	0.0433	0.0266	0.039	0.0301	0.0434	0.0287	0.0275
47	0.0487	0.0327	0.0191	0.0225	0.0191	0.0279	0.0192	0.0195
48	0.0276	0.0275	0.0123	0.0156	0.0105	0.0105	0.0104	0.0102
49	0.0441	0.0286	0.0279	0.034	0.0278	0.0301	0.0287	0.027
50	0.015	0.0067	0.0113	0.014	0.0122	0.0225	0.0115	0.0111
51	0.0878	0.0872	0.0501	0.0754	0.0513	0.0826	0.0497	0.0481
52	0.0186	0.022	0.0212	0.0217	0.0194	0.0384	0.0175	0.0163
53	0.0318	0.0213	0.0265	0.0339	0.0182	0.0217	0.0171	0.0161
54	0.0503	0.0334	0.0239	0.029	0.0205	0.0304	0.0191	0.0185
55	0.0508	0.0204	0.0172	0.0136	0.0104	0.0258	0.0102	0.0104
56	0.0537	0.0178	0.0204	0.0176	0.0126	0.0272	0.0127	0.0128
57	0.0296	0.0164	0.0147	0.0243	0.0174	0.0264	0.016	0.0152
58	0.0328	0.0103	0.0064	0.0099	0.0063	0.0121	0.0063	0.0057
59	0.0336	0.0249	0.0145	0.0152	0.016	0.02	0.0158	0.0153
60	0.0248	0.0098	0.0168	0.0216	0.011	0.02	0.0098	0.0092
61	0.077	0.0284	0.0227	0.0257	0.0283	0.0479	0.0264	0.0251
62	0.0224	0.0083	0.0055	0.004	0.0059	0.0076	0.006	0.0059
63	0.0378	0.0319	0.0142	0.0147	0.0141	0.0187	0.0143	0.0142
64	0.0163	0.0098	0.0036	0.0057	0.0029	0.0027	0.0031	0.003
65	0.0873	0.0113	0.0101	0.018	0.0137	0.0308	0.0124	0.0112
66	0.022	0.0145	0.0152	0.0144	0.0127	0.0254	0.0125	0.0125
67	0.0251	0.0299	0.007	0.0054	0.0057	0.0087	0.0053	0.0063
68	0.0276	0.022	0.0154	0.0169	0.0163	0.015	0.0162	0.016
69	0.0417	0.024	0.013	0.0183	0.0103	0.0135	0.0103	0.0098
70	0.0595	0.0296	0.0327	0.0396	0.0312	0.0499	0.0294	0.0282
71	0.0696	0.047	0.0274	0.0327	0.0292	0.0541	0.0286	0.0284
72	0.0301	0.0279	0.0095	0.008	0.0087	0.0128	0.0085	0.009
73	0.0292	0.0183	0.0191	0.0269	0.0166	0.0193	0.0166	0.0157
74	0.0387	0.0349	0.03	0.0406	0.0261	0.0277	0.0256	0.0256
75	0.0334	0.0214	0.0595	0.0768	0.0442	0.0514	0.0406	0.0391
76	0.0413	0.0085	0.014	0.0092	0.0069	0.0158	0.0074	0.0078

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
77	0.0318	0.0246	0.011	0.014	0.01	0.0149	0.0092	0.0098
78	0.0373	0.0147	0.0265	0.0229	0.0222	0.0256	0.0214	0.0202
79	0.0284	0.0291	0.0276	0.0378	0.0233	0.0252	0.0224	0.0224
80	0.0487	0.0303	0.0273	0.0482	0.0289	0.0452	0.0279	0.0263
81	0.0469	0.0385	0.0431	0.0565	0.0322	0.0395	0.0305	0.0292
82	0.0359	0.0162	0.0219	0.0364	0.0131	0.0113	0.0122	0.0105
83	0.0269	0.0347	0.0082	0.0127	0.0077	0.0103	0.0079	0.0075
84	0.0195	0.0085	0.0053	0.007	0.0052	0.0057	0.0057	0.0053
85	0.0096	0.0063	0.0108	0.0069	0.0047	0.0066	0.0035	0.0036
86	0.0932	0.0528	0.0306	0.0438	0.025	0.0363	0.0244	0.0242
87	0.0416	0.0266	0.0247	0.0351	0.019	0.0251	0.018	0.0171
88	0.0214	0.0158	0.0144	0.0185	0.0135	0.0136	0.0136	0.0125
89	0.0397	0.0431	0.0187	0.0172	0.0156	0.0244	0.0155	0.0157
90	0.0162	0.0102	0.0071	0.0057	0.0093	0.0134	0.0059	0.0053
91	0.0577	0.0269	0.0386	0.0596	0.0291	0.0354	0.0276	0.0263
92	0.0361	0.0228	0.0146	0.0172	0.013	0.0296	0.0113	0.0109
93	0.0097	0.0156	0.0186	0.0213	0.0142	0.0136	0.0121	0.0111
94	0.0178	0.0174	0.0169	0.0196	0.0118	0.0165	0.009	0.0082
95	0.0081	0.01	0.0097	0.0099	0.0118	0.0148	0.007	0.0067
96	0.044	0.018	0.0294	0.0461	0.0252	0.0352	0.0198	0.0183
97	0.0158	0.0148	0.0111	0.009	0.0093	0.0126	0.0089	0.0087
98	0.0085	0.0057	0.0057	0.0063	0.0026	0.0023	0.0024	0.0023
99	0.012	0.0055	0.0102	0.01	0.0059	0.0072	0.0055	0.0056
100	0.0106	0.0094	0.0069	0.0074	0.0103	0.0141	0.0051	0.0045
101	0.0393	0.0259	0.014	0.0199	0.012	0.0289	0.0105	0.0105
102	0.0386	0.027	0.0133	0.0189	0.0143	0.0146	0.0149	0.0142
103	0.02	0.0236	0.0101	0.0124	0.011	0.0113	0.0115	0.0106
104	0.0091	0.011	0.0035	0.0041	0.0021	0.0026	0.0022	0.002
105	0.0069	0.0074	0.0101	0.0108	0.0095	0.0114	0.006	0.0054
106	0.0187	0.0127	0.0142	0.0172	0.0117	0.0228	0.0112	0.0114
107	0.0324	0.0183	0.0148	0.0212	0.0125	0.0107	0.0123	0.0119
108	0.021	0.0229	0.0151	0.0212	0.0159	0.0129	0.0165	0.0153
109	0.0191	0.0129	0.013	0.0184	0.0158	0.0111	0.0166	0.0157
110	0.0269	0.011	0.0118	0.0135	0.0105	0.0147	0.007	0.0064
111	0.0495	0.0172	0.0218	0.0379	0.0192	0.0408	0.0167	0.0156
112	0.0374	0.0221	0.0328	0.0423	0.0243	0.0196	0.0229	0.0225
113	0.0136	0.0159	0.0171	0.0267	0.0149	0.0117	0.0154	0.0141
114	0.0341	0.0212	0.0167	0.0203	0.0173	0.0151	0.0171	0.0165
115	0.0246	0.0103	0.0165	0.0193	0.015	0.0187	0.0117	0.0113
116	0.014	0.0079	0.0241	0.0446	0.0176	0.0229	0.0147	0.0134

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
117	0.0412	0.0477	0.0274	0.0287	0.0242	0.0232	0.0239	0.0246
118	0.0157	0.023	0.0391	0.0446	0.0258	0.0183	0.0234	0.023
119	0.0333	0.0062	0.0198	0.0273	0.0132	0.0175	0.0099	0.0088
120	0.0154	0.0077	0.0123	0.0181	0.0121	0.0195	0.0074	0.0065
121	0.0136	0.0145	0.0104	0.0117	0.0131	0.0198	0.0099	0.0099
122	0.0071	0.0104	0.0118	0.0078	0.0056	0.0092	0.0057	0.0062
123	0.0145	0.011	0.0106	0.0134	0.0096	0.0104	0.0094	0.0088
124	0.0094	0.0098	0.0123	0.0113	0.0054	0.0058	0.0057	0.0062
125	0.0178	0.0121	0.0262	0.0402	0.0201	0.0269	0.0164	0.0156
126	0.0112	0.0098	0.0078	0.006	0.0077	0.0106	0.0052	0.005
127	0.0304	0.0266	0.016	0.0141	0.0132	0.0148	0.0133	0.0135
128	0.0206	0.0161	0.007	0.0077	0.0079	0.0073	0.0084	0.0079
129	0.0159	0.0282	0.012	0.0092	0.01	0.014	0.0093	0.0105
130	0.0182	0.008	0.0345	0.0474	0.0241	0.0257	0.0195	0.0178
131	0.0186	0.0149	0.0139	0.0141	0.0105	0.017	0.0088	0.0089
132	0.0108	0.0158	0.0091	0.0088	0.0096	0.0105	0.0098	0.0098
133	0.0116	0.02	0.0097	0.0122	0.0105	0.0091	0.0108	0.0106
134	0.0133	0.0283	0.0066	0.0085	0.007	0.0092	0.0072	0.0078
135	0.0157	0.0095	0.0246	0.0331	0.0166	0.0204	0.0145	0.0144
136	0.0251	0.0106	0.01	0.0071	0.0066	0.0114	0.0069	0.0076
137	0.0099	0.0148	0.0063	0.0106	0.0048	0.0072	0.0047	0.0045
138	0.0252	0.0189	0.0151	0.0197	0.0125	0.0122	0.0125	0.0126
139	0.023	0.0117	0.0303	0.0427	0.0201	0.0173	0.0187	0.0178
140	0.0147	0.0107	0.0153	0.0226	0.0108	0.0141	0.0102	0.0105
141	0.0377	0.023	0.0131	0.0277	0.0242	0.0424	0.0184	0.017
142	0.0444	0.0251	0.0324	0.0445	0.0306	0.0442	0.0253	0.0226
143	0.0353	0.0125	0.054	0.0644	0.0384	0.0336	0.0339	0.0319
144	0.0357	0.0185	0.0358	0.0426	0.0238	0.018	0.0217	0.0212
145	0.0277	0.0229	0.0393	0.0403	0.0289	0.02	0.0307	0.0316
146	0.0498	0.046	0.0143	0.0162	0.0148	0.0132	0.0162	0.017
147	0.0155	0.0091	0.0103	0.0155	0.0051	0.0048	0.0056	0.005
148	0.0088	0.0249	0.0115	0.0131	0.0114	0.0131	0.0119	0.0118
149	0.0129	0.012	0.0065	0.006	0.0073	0.0099	0.007	0.0071
150	0.0014	0.0018	0.0028	0.0015	0.0026	0.0019	0.0009	0.0009
151	0.0216	0.0392	0.0257	0.0321	0.0209	0.0198	0.0221	0.0221
152	0.0179	0.0133	0.0127	0.0178	0.0079	0.0075	0.0082	0.0077
153	0.0114	0.0092	0.0067	0.0078	0.0071	0.0066	0.0069	0.0071
154	0.0096	0.0114	0.0065	0.0056	0.0088	0.0092	0.0081	0.0079
155	0.0017	0.0015	0.0031	0.0011	0.0018	0.0011	0.0005	0.0006
156	0.017	0.025	0.017	0.0256	0.0124	0.0119	0.0131	0.0127

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
157	0.0061	0.0038	0.0118	0.0158	0.0078	0.0038	0.0079	0.007
158	0.0103	0.007	0.0088	0.0091	0.0071	0.0071	0.005	0.0046
159	0.0155	0.0087	0.0066	0.0083	0.0101	0.0081	0.0067	0.0059
160	0.0026	0.0028	0.0029	0.0013	0.0033	0.0015	0.001	0.001
161	0.0119	0.003	0.0071	0.0101	0.0049	0.0047	0.004	0.0043
162	0.0077	0.0074	0.0061	0.0068	0.007	0.0074	0.0044	0.0043
163	0.0074	0.0039	0.0035	0.0025	0.0028	0.0038	0.0024	0.0023
164	0.0032	0.0021	0.0035	0.0026	0.0019	0.0035	0.0011	0.0011
165	0.0036	0.0044	0.0034	0.0033	0.0051	0.0028	0.0016	0.0016
166	0.0125	0.0071	0.0233	0.034	0.0179	0.0117	0.0157	0.0137
167	0.0029	0.0111	0.0032	0.0029	0.0036	0.0064	0.0041	0.0048
168	0.006	0.0047	0.0024	0.0018	0.0018	0.0039	0.002	0.0019
169	0.0023	0.0027	0.0029	0.0023	0.0021	0.0039	0.0019	0.0019
170	0.0057	0.0039	0.0023	0.0034	0.0049	0.0018	0.0016	0.0014
171	0.0134	0.0047	0.0124	0.0184	0.0089	0.0054	0.008	0.0076
172	0.0185	0.0225	0.0127	0.0162	0.0106	0.012	0.0103	0.0101
173	0.0161	0.0084	0.0093	0.0122	0.0081	0.0078	0.0087	0.0082
174	0.0076	0.0047	0.0023	0.0026	0.002	0.0042	0.0022	0.0024
175	0.0026	0.0054	0.0044	0.0054	0.0062	0.0039	0.0024	0.0023
176	0.0131	0.0077	0.0119	0.0146	0.008	0.0071	0.0086	0.009
177	0.0274	0.0323	0.0108	0.0142	0.01	0.0112	0.0111	0.0115
178	0.0093	0.0291	0.0092	0.0101	0.0119	0.0117	0.0125	0.013
179	0.0078	0.0052	0.0088	0.0112	0.0068	0.0048	0.0078	0.0075
180	0.0068	0.0057	0.0061	0.0092	0.0077	0.0058	0.0035	0.0034
181	0.0097	0.0167	0.0085	0.0117	0.008	0.0074	0.0094	0.0104
182	0.0202	0.0296	0.0254	0.0306	0.0164	0.0148	0.0174	0.0166
183	0.0229	0.0159	0.0233	0.0278	0.0168	0.0105	0.0181	0.017
184	0.0084	0.0137	0.0231	0.0237	0.0168	0.0131	0.0165	0.0158
185	0.0134	0.0132	0.0127	0.0191	0.015	0.0137	0.0087	0.0082
186	0.0036	0.0051	0.0015	0.0011	0.005	0.0019	0.001	0.0009
187	0.0042	0.0024	0.0062	0.0047	0.0017	0.0051	0.0018	0.0022
188	0.0024	0.0016	0.0039	0.0033	0.0031	0.0029	0.0025	0.0023
189	0.0035	0.0029	0.0047	0.0044	0.0039	0.0038	0.0029	0.0031
190	0.0212	0.0066	0.0104	0.0178	0.0118	0.0083	0.0116	0.0116
191	0.0064	0.0072	0.0063	0.0071	0.0066	0.0061	0.0032	0.0034
192	0.0138	0.0066	0.0069	0.0059	0.0051	0.0077	0.0052	0.0052
193	0.0041	0.0116	0.0062	0.0053	0.0048	0.0093	0.0049	0.0055
194	0.0051	0.003	0.004	0.0066	0.0039	0.0023	0.0042	0.0038
195	0.0117	0.0053	0.012	0.0169	0.0086	0.0075	0.0076	0.0072
196	0.0034	0.005	0.0031	0.0051	0.0065	0.0034	0.0024	0.002

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
197	0.0146	0.0123	0.0062	0.0052	0.0075	0.0104	0.0073	0.0079
198	0.0046	0.0192	0.0054	0.0037	0.0066	0.0117	0.0059	0.0068
199	0.0175	0.0111	0.0109	0.0147	0.0089	0.0068	0.0099	0.0096
200	0.0068	0.0125	0.013	0.0163	0.0103	0.0104	0.0114	0.0117
201	0.0123	0.0099	0.0058	0.0089	0.0084	0.0086	0.0044	0.0044
202	0.0129	0.0086	0.009	0.0106	0.0081	0.0084	0.0077	0.0077
203	0.008	0.0186	0.016	0.0175	0.0145	0.0128	0.0145	0.015
204	0.0109	0.0213	0.0273	0.034	0.0201	0.0151	0.0212	0.021
205	0.0088	0.009	0.0161	0.0197	0.0119	0.0095	0.012	0.0114
206	0.0204	0.0099	0.0068	0.0109	0.0091	0.0084	0.0034	0.0032
207	0.0149	0.0067	0.0116	0.0137	0.0089	0.0069	0.0095	0.0094
208	0.0247	0.0262	0.0202	0.0226	0.0181	0.0155	0.0199	0.0203
209	0.0171	0.0381	0.0345	0.0386	0.0262	0.0214	0.0276	0.0282
210	0.012	0.0075	0.0228	0.0329	0.0149	0.0096	0.0157	0.0158
211	0.0047	0.0083	0.1786	0.0061	0.0052	0.0069	0.0053	0.0051
212	0.0026	0.0052	0.0154	0.0029	0.0024	0.006	0.0023	0.0022
213	0.0054	0.0033	0.0102	0.0041	0.0034	0.0044	0.0036	0.0035
214	0.0015	0.0007	0.0145	0.0007	0.0006	0.0033	0.0006	0.0007
215	0.0003	0.0003	0.1212	0.0004	0.0004	0.0031	0.0002	0.0003
216	0.0114	0.0097	0.1431	0.0022	0.003	0.0069	0.0025	0.0025
217	0.0041	0.0036	0.012	0.0033	0.0025	0.0046	0.0026	0.0024
218	0.0032	0.0047	0.0209	0.003	0.003	0.0058	0.0027	0.0027
219	0.0029	0.0011	0.0276	0.001	0.0012	0.0034	0.001	0.0009
220	0.0004	0.0003	0.076	0.0001	0.0004	0.002	0.0001	0.0001
221	0.0086	0.0097	0.1198	0.0043	0.0043	0.0081	0.0039	0.004
222	0.0024	0.0025	0.0651	0.001	0.0011	0.0043	0.0011	0.0011
223	0.0023	0.0017	0.0344	0.0027	0.0022	0.0028	0.002	0.002
224	0.0022	0.0022	0.0594	0.0026	0.0026	0.0037	0.0021	0.0019
225	0.0006	0.0008	0.173	0.0003	0.001	0.0017	0.0002	0.0001
226	0.0019	0.002	0.1958	0.0043	0.003	0.0021	0.0026	0.0025
227	0.0024	0.0006	0.1381	0.0008	0.0009	0.0022	0.0006	0.0007
228	0.0009	0.0005	0.04	0.0006	0.0004	0.0038	0.0003	0.0005
229	0.0004	0.0003	0.0456	0.0004	0.0004	0.0029	0.0002	0.0002
230	0.0004	0.0021	0.1307	0.0015	0.0027	0.0012	0.0006	0.0006
231	0.0031	0.0023	0.081	0.0027	0.0021	0.0036	0.0023	0.0024
232	0.003	0.0027	0.0167	0.0019	0.0022	0.0051	0.0021	0.0024
233	0.0014	0.0018	0.0099	0.0017	0.0015	0.0048	0.0015	0.0015
234	0.0007	0.0009	0.0289	0.0009	0.0008	0.004	0.0008	0.0009
235	0.0027	0.0023	0.1395	0.0026	0.0029	0.0026	0.0018	0.0017
236	0.0032	0.0018	0.0711	0.0015	0.0011	0.0037	0.0012	0.0013

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
237	0.0057	0.006	0.0273	0.0044	0.0041	0.0072	0.0043	0.0042
238	0.0022	0.0066	0.0072	0.0019	0.0031	0.0079	0.0027	0.0029
239	0.0035	0.0027	0.0224	0.0021	0.0024	0.006	0.0022	0.0024
240	0.0007	0.0012	0.1027	0.0011	0.0016	0.0008	0.0004	0.0003
241	0.0028	0.0034	0.0815	0.0019	0.0018	0.0052	0.0017	0.0018
242	0.0021	0.0053	0.0213	0.002	0.0022	0.0066	0.0019	0.0021
243	0.006	0.0062	0.0209	0.0039	0.0044	0.0085	0.0042	0.0042
244	0.0027	0.0012	0.0286	0.0014	0.001	0.0039	0.0011	0.0011
245	0.0018	0.0032	0.1674	0.0037	0.0038	0.0025	0.0022	0.0021
246	0.004	0.0045	0.3311	0.0028	0.0025	0.0054	0.0025	0.0026
247	0.0092	0.0144	0.0326	0.0038	0.0045	0.008	0.004	0.0041
248	0.014	0.0088	0.0799	0.008	0.0072	0.0096	0.0075	0.0072
249	0.0025	0.0032	0.0367	0.0048	0.0036	0.0044	0.0039	0.0035
250	0.0021	0.0032	0.1178	0.0049	0.0044	0.0031	0.0031	0.0029
251	0.0013	0.002	0.3554	0.0016	0.0025	0.0017	0.0008	0.0008
252	0.0009	0.0007	0.0632	0.0006	0.0009	0.0024	0.0007	0.0007
253	0.0013	0.0008	0.0344	0.0009	0.001	0.0037	0.0009	0.0009
254	0.0014	0.0008	0.0287	0.001	0.0013	0.0018	0.0007	0.0007
255	0.0056	0.0032	0.2037	0.0031	0.0029	0.007	0.0042	0.0046
256	0.0011	0.0015	0.2108	0.0013	0.0022	0.0006	0.0005	0.0004
257	0.0014	0.0012	0.0358	0.0009	0.0011	0.0048	0.0012	0.0013
258	0.0018	0.0012	0.0202	0.0007	0.0011	0.0038	0.001	0.0011
259	0.004	0.0026	0.0104	0.0026	0.0024	0.0039	0.0021	0.0021
260	0.0029	0.0015	0.0971	0.0015	0.0013	0.0038	0.0013	0.0015
261	0.0023	0.0019	0.1107	0.0019	0.0021	0.0018	0.0014	0.0014
262	0.0028	0.0019	0.0155	0.0018	0.0017	0.0044	0.0017	0.0018
263	0.0056	0.0029	0.0098	0.0008	0.0015	0.0062	0.0012	0.0014
264	0.0046	0.0064	0.0591	0.005	0.0046	0.0074	0.0049	0.0048
265	0.0052	0.0051	0.1054	0.0064	0.0053	0.0063	0.0055	0.0054
266	0.0025	0.0009	0.1056	0.0013	0.0013	0.0018	0.0008	0.0008
267	0.0037	0.0011	0.0986	0.0014	0.001	0.0041	0.0011	0.0013
268	0.0033	0.0049	0.0219	0.0033	0.0033	0.0062	0.003	0.003
269	0.0113	0.0074	0.0336	0.0086	0.007	0.0086	0.0073	0.007
270	0.007	0.0052	0.0525	0.0074	0.0054	0.0063	0.0058	0.0056
271	0.0012	0.0008	0.1981	0.001	0.0009	0.002	0.0011	0.0014
272	0.0132	0.0041	0.1021	0.0043	0.0037	0.0056	0.004	0.0039
273	0.0033	0.004	0.0317	0.0037	0.0026	0.0042	0.0028	0.0026
274	0.0059	0.008	0.036	0.0079	0.0057	0.0077	0.0061	0.0059
275	0.0023	0.0042	0.3323	0.0072	0.0045	0.0051	0.0049	0.0044

<u>Sample Size 9</u>

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
1	0.0968	0.0711	0.026	0.0295	0.0245	0.0609	0.0238	0.0231
2	0.0245	0.0342	0.0175	0.0177	0.0172	0.017	0.0174	0.018
3	0.0258	0.0269	0.0218	0.026	0.0222	0.0209	0.0226	0.0166
4	0.0198	0.0327	0.0148	0.0183	0.0135	0.016	0.014	0.0105
5	0.0149	0.0311	0.0412	0.0661	0.0563	0.0655	0.0551	0.0467
6	0.0832	0.0491	0.0408	0.0716	0.0677	0.103	0.0669	0.0521
7	0.0762	0.0632	0.0462	0.043	0.0477	0.0671	0.0478	0.0447
8	0.0443	0.0425	0.0114	0.0113	0.0122	0.0163	0.0121	0.0116
9	0.0328	0.0257	0.0353	0.0399	0.0307	0.0323	0.0305	0.027
10	0.0728	0.0351	0.0188	0.0293	0.0288	0.0496	0.0282	0.0221
11	0.1308	0.0887	0.0228	0.036	0.038	0.0698	0.0383	0.0337
12	0.0828	0.102	0.0514	0.0532	0.0484	0.0832	0.0464	0.0388
13	0.0633	0.0235	0.0157	0.0176	0.0153	0.0234	0.0144	0.0141
14	0.0814	0.0244	0.0371	0.0402	0.0333	0.0545	0.0314	0.0245
15	0.0457	0.0329	0.0275	0.0285	0.0181	0.0131	0.018	0.0191
16	0.0624	0.0386	0.0449	0.0592	0.0384	0.0372	0.0362	0.0357
17	0.0224	0.0245	0.02	0.0258	0.0173	0.0207	0.0173	0.0161
18	0.0311	0.0264	0.0315	0.0413	0.0306	0.0266	0.0309	0.0278
19	0.0393	0.0365	0.0107	0.0128	0.0121	0.0171	0.0125	0.0124
20	0.0175	0.0153	0.0178	0.0236	0.0167	0.022	0.014	0.012
21	0.0664	0.0481	0.0704	0.0953	0.0617	0.0715	0.0593	0.0504
22	0.0888	0.0517	0.0444	0.051	0.0457	0.0443	0.0463	0.0453
23	0.0285	0.0205	0.0173	0.024	0.0153	0.0177	0.0159	0.0123
24	0.0478	0.0377	0.0144	0.0185	0.0139	0.0206	0.0143	0.0139
25	0.0854	0.0238	0.0292	0.0403	0.0319	0.0465	0.0271	0.0229
26	0.0659	0.0512	0.0638	0.0898	0.0645	0.078	0.0634	0.0578
27	0.1157	0.0566	0.0336	0.0401	0.0321	0.0697	0.0292	0.0235
28	0.0679	0.0234	0.0209	0.0266	0.0167	0.0148	0.0169	0.0172
29	0.066	0.0607	0.0292	0.0312	0.0311	0.0574	0.0295	0.0272
30	0.0317	0.0139	0.0143	0.0214	0.0146	0.0296	0.0112	0.0085
31	0.1266	0.0716	0.0655	0.0897	0.0589	0.0878	0.0551	0.0477
32	0.0354	0.0183	0.0176	0.0257	0.0187	0.0337	0.0164	0.0125
33	0.0262	0.0325	0.0402	0.0407	0.0359	0.039	0.0336	0.0311
34	0.0582	0.0251	0.0232	0.026	0.019	0.0324	0.017	0.0144
35	0.0731	0.0498	0.0528	0.0637	0.0421	0.0637	0.0416	0.0338
36	0.0568	0.0227	0.0198	0.0331	0.0264	0.0416	0.0248	0.0199
37	0.0483	0.037	0.011	0.0129	0.0132	0.0154	0.0138	0.0141
38	0.0599	0.0471	0.0427	0.0497	0.0394	0.0393	0.0403	0.0339

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
39	0.0366	0.0571	0.0258	0.0261	0.0274	0.0302	0.0274	0.0281
40	0.0516	0.0181	0.0093	0.0145	0.0151	0.0348	0.0124	0.008
41	0.1341	0.0358	0.0571	0.0829	0.0599	0.0704	0.0561	0.0506
42	0.0326	0.0386	0.0177	0.0189	0.0165	0.0206	0.0167	0.0149
43	0.0765	0.0198	0.021	0.0296	0.027	0.0216	0.0289	0.0203
44	0.0469	0.019	0.0117	0.0173	0.0141	0.0137	0.0152	0.0106
45	0.0258	0.0057	0.033	0.037	0.0349	0.0208	0.0314	0.025
46	0.1272	0.0583	0.0532	0.0732	0.0612	0.0968	0.0596	0.0535
47	0.0731	0.0396	0.0329	0.0368	0.0292	0.0258	0.0285	0.0248
48	0.0367	0.0514	0.0331	0.037	0.031	0.0391	0.0303	0.0319
49	0.0351	0.0298	0.0228	0.0253	0.018	0.0284	0.0173	0.0184
50	0.057	0.0283	0.0494	0.0592	0.0466	0.0535	0.0428	0.0436
51	0.0334	0.0253	0.0724	0.116	0.0903	0.0732	0.0873	0.0763
52	0.1041	0.0537	0.0159	0.012	0.0204	0.0432	0.0198	0.0162
53	0.0665	0.0331	0.0081	0.009	0.0119	0.0132	0.0129	0.0158
54	0.0478	0.0342	0.0347	0.0499	0.0373	0.0482	0.0354	0.0306
55	0.0533	0.0335	0.0339	0.0407	0.029	0.0374	0.0302	0.0331
56	0.053	0.0331	0.0766	0.1046	0.0879	0.0901	0.0854	0.0751
57	0.0419	0.0247	0.0112	0.0169	0.0206	0.0372	0.0175	0.0137
58	0.0428	0.0414	0.0231	0.0307	0.025	0.0363	0.0229	0.0207
59	0.0476	0.0647	0.0253	0.0309	0.0223	0.0304	0.0217	0.0246
60	0.1276	0.0724	0.1257	0.1438	0.1111	0.103	0.1182	0.1102
61	0.0293	0.0118	0.004	0.0132	0.0168	0.0358	0.0132	0.0076
62	0.0682	0.0612	0.036	0.0319	0.0419	0.0463	0.0413	0.0381
63	0.0382	0.029	0.0222	0.0242	0.0205	0.025	0.0208	0.0206
64	0.1155	0.0482	0.031	0.0395	0.0248	0.0381	0.0237	0.0226
65	0.0891	0.0459	0.0276	0.0412	0.0245	0.0369	0.0224	0.0224
66	0.0603	0.0198	0.0221	0.0213	0.019	0.0286	0.0181	0.019
67	0.0523	0.042	0.0266	0.0311	0.0282	0.0271	0.0285	0.0304
68	0.0468	0.0238	0.0114	0.0136	0.0118	0.013	0.0122	0.0132
69	0.0887	0.0627	0.0333	0.0428	0.0292	0.0402	0.0291	0.0287
70	0.0967	0.0311	0.0489	0.0499	0.0388	0.0351	0.0378	0.0386
71	0.0403	0.0291	0.0191	0.026	0.0194	0.0239	0.0174	0.0171
72	0.0154	0.0055	0.0116	0.0133	0.0154	0.0136	0.0163	0.0133
73	0.0276	0.0416	0.0334	0.0382	0.031	0.026	0.0308	0.0297
74	0.0446	0.0475	0.0347	0.0373	0.0307	0.0353	0.0303	0.0268
75	0.0971	0.0795	0.1367	0.1703	0.1172	0.1438	0.1106	0.0917
76	0.1096	0.0453	0.0532	0.0775	0.0563	0.0813	0.0553	0.0516
77	0.0212	0.0114	0.0163	0.0208	0.0177	0.0261	0.0165	0.0139
78	0.1055	0.0847	0.0973	0.1127	0.076	0.1135	0.0699	0.0623

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
79	0.139	0.0467	0.0537	0.0579	0.0483	0.0345	0.0454	0.0415
80	0.1299	0.0482	0.0539	0.0779	0.0472	0.0685	0.0444	0.0337
81	0.1333	0.0413	0.0459	0.0658	0.034	0.0393	0.0338	0.0291
82	0.0212	0.0246	0.0272	0.0364	0.0213	0.016	0.0197	0.0185
83	0.0482	0.0176	0.0186	0.0238	0.0182	0.0135	0.0186	0.017
84	0.0209	0.0315	0.0046	0.0037	0.0046	0.0099	0.0045	0.0071
85	0.0151	0.0185	0.0076	0.0098	0.0175	0.021	0.012	0.0081
86	0.0691	0.0621	0.0646	0.0856	0.0484	0.0525	0.0457	0.0408
87	0.0079	0.018	0.0145	0.0241	0.0109	0.0073	0.0108	0.0069
88	0.0528	0.0487	0.0503	0.0601	0.0438	0.0352	0.044	0.0406
89	0.0208	0.0149	0.019	0.0217	0.014	0.0104	0.0144	0.0151
90	0.0188	0.0173	0.0222	0.0229	0.0188	0.0212	0.0174	0.0187
91	0.0652	0.0487	0.0589	0.0868	0.0405	0.0315	0.038	0.0308
92	0.0649	0.0295	0.0263	0.0361	0.028	0.0496	0.0251	0.022
93	0.0302	0.047	0.028	0.0236	0.0249	0.0265	0.0233	0.0252
94	0.027	0.0158	0.0152	0.0158	0.0172	0.0163	0.0139	0.0104
95	0.035	0.0303	0.0151	0.0236	0.0268	0.0376	0.0195	0.0192
96	0.0132	0.024	0.0323	0.0452	0.0255	0.0397	0.0213	0.0206
97	0.0422	0.0207	0.0175	0.0184	0.0239	0.0228	0.0208	0.019
98	0.0212	0.0162	0.0104	0.0108	0.0077	0.0093	0.0064	0.0088
99	0.0378	0.0211	0.0108	0.0115	0.0159	0.0192	0.0143	0.0113
100	0.015	0.0162	0.0337	0.0424	0.0291	0.0274	0.0211	0.0191
101	0.043	0.0071	0.0125	0.0154	0.0068	0.0106	0.0058	0.0067
102	0.0255	0.0233	0.0142	0.0144	0.014	0.0128	0.0138	0.0151
103	0.0208	0.0391	0.0261	0.032	0.033	0.0236	0.0352	0.0319
104	0.0122	0.0146	0.0116	0.011	0.01	0.0086	0.0097	0.0093
105	0.0075	0.0064	0.0136	0.0139	0.0106	0.0109	0.0089	0.0093
106	0.0311	0.0221	0.0354	0.0458	0.0335	0.0417	0.0305	0.0251
107	0.0491	0.0195	0.0221	0.029	0.0226	0.0229	0.0229	0.0195
108	0.0316	0.038	0.0255	0.0277	0.0242	0.0231	0.0233	0.0237
109	0.0599	0.051	0.0351	0.0398	0.0342	0.0441	0.0348	0.0275
110	0.0093	0.0086	0.0161	0.0174	0.0099	0.0098	0.0072	0.0085
111	0.0616	0.0347	0.0597	0.0925	0.0497	0.0593	0.0424	0.0325
112	0.0654	0.0786	0.0533	0.0634	0.0432	0.0467	0.0442	0.0418
113	0.0158	0.0278	0.0227	0.0316	0.0164	0.0107	0.0158	0.0143
114	0.0313	0.0269	0.0294	0.0354	0.0244	0.0208	0.0239	0.0213
115	0.0647	0.0262	0.0318	0.0356	0.0257	0.0345	0.0228	0.0219
116	0.1091	0.0467	0.0692	0.0859	0.0586	0.0605	0.0599	0.0577
117	0.0548	0.0736	0.0776	0.1018	0.0633	0.0563	0.0593	0.0512
118	0.061	0.0477	0.0702	0.0781	0.0522	0.0487	0.0519	0.0465

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
119	0.0666	0.0354	0.075	0.0884	0.0543	0.0462	0.0512	0.0414
120	0.074	0.0466	0.0474	0.0646	0.0519	0.0534	0.0476	0.0396
121	0.0343	0.0167	0.0237	0.0214	0.0144	0.0179	0.0135	0.0166
122	0.0144	0.0086	0.0107	0.0152	0.0152	0.0168	0.0108	0.0084
123	0.0134	0.0061	0.0041	0.0038	0.0049	0.0067	0.0045	0.0045
124	0.0183	0.0167	0.0174	0.018	0.0112	0.0121	0.0106	0.0131
125	0.0931	0.028	0.0342	0.0405	0.0275	0.0382	0.027	0.0279
126	0.0602	0.037	0.0274	0.0394	0.0413	0.0481	0.0313	0.0288
127	0.0323	0.0259	0.0193	0.0182	0.0185	0.0199	0.0179	0.0185
128	0.0257	0.0344	0.0118	0.0137	0.0133	0.0169	0.0138	0.0147
129	0.0384	0.0229	0.0347	0.0451	0.0362	0.0225	0.036	0.0281
130	0.0372	0.0185	0.0487	0.0558	0.0333	0.0235	0.0333	0.0313
131	0.0673	0.028	0.0146	0.0218	0.0232	0.0353	0.0166	0.0151
132	0.0587	0.0466	0.0318	0.0316	0.0373	0.0347	0.0369	0.0366
133	0.0387	0.0267	0.0152	0.0202	0.0162	0.011	0.018	0.0159
134	0.0356	0.0378	0.0327	0.0437	0.027	0.0229	0.0272	0.0251
135	0.0537	0.0411	0.0247	0.0314	0.0226	0.0375	0.0206	0.0191
136	0.0262	0.0097	0.0071	0.0111	0.0104	0.0145	0.0074	0.0066
137	0.0116	0.0295	0.0259	0.0284	0.0275	0.0269	0.0251	0.0268
138	0.0588	0.0188	0.0209	0.0313	0.0162	0.0147	0.0162	0.0136
139	0.0376	0.0146	0.0217	0.0359	0.0132	0.011	0.0125	0.0102
140	0.2623	0.0576	0.1452	0.1789	0.1043	0.1061	0.0974	0.0813
141	0.0311	0.028	0.0294	0.0442	0.0316	0.0415	0.0259	0.0225
142	0.0644	0.0327	0.0307	0.0346	0.0247	0.0291	0.0234	0.0228
143	0.041	0.0453	0.0989	0.1171	0.0732	0.066	0.0683	0.0587
144	0.0893	0.0733	0.1035	0.1102	0.0783	0.0684	0.0781	0.0761
145	0.1168	0.0606	0.1112	0.1239	0.0865	0.0703	0.092	0.0832
146	0.0358	0.0393	0.0267	0.0359	0.0186	0.0146	0.0207	0.0171
147	0.0111	0.0189	0.0138	0.0192	0.0119	0.0083	0.0132	0.0123
148	0.0315	0.0218	0.0107	0.0123	0.0099	0.0147	0.0095	0.0103
149	0.0227	0.0109	0.0047	0.0052	0.007	0.0092	0.0067	0.0069
150	0.0019	0.0028	0.0054	0.0032	0.0031	0.0042	0.0013	0.0031
151	0.0473	0.0502	0.0246	0.0379	0.0318	0.0296	0.0327	0.0315
152	0.0545	0.0346	0.0102	0.0124	0.0117	0.0127	0.0125	0.0157
153	0.0532	0.0291	0.0172	0.0178	0.0212	0.0207	0.0214	0.0214
154	0.013	0.0093	0.0075	0.0054	0.0045	0.0097	0.0046	0.0073
155	0.0053	0.0058	0.0037	0.0029	0.0062	0.0041	0.0024	0.002
156	0.0489	0.0242	0.0197	0.0261	0.0159	0.0131	0.0182	0.0184
157	0.0251	0.012	0.0104	0.0127	0.007	0.0069	0.0075	0.0076
158	0.0169	0.0117	0.0157	0.0199	0.0129	0.0124	0.0138	0.014

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
159	0.009	0.011	0.0158	0.0133	0.0111	0.0148	0.0116	0.0145
160	0.0019	0.003	0.0034	0.0015	0.0038	0.0017	0.0008	0.0019
161	0.0107	0.0064	0.0155	0.0258	0.0151	0.0127	0.0099	0.0099
162	0.0161	0.011	0.0099	0.0097	0.0071	0.0088	0.0062	0.0075
163	0.0032	0.0041	0.0059	0.0059	0.0069	0.0041	0.0054	0.0044
164	0.0054	0.0034	0.0046	0.0027	0.0035	0.0044	0.003	0.0036
165	0.0012	0.0064	0.0052	0.0046	0.0063	0.0055	0.003	0.004
166	0.0283	0.0078	0.0126	0.019	0.0096	0.0093	0.0083	0.0086
167	0.024	0.0143	0.0121	0.0133	0.0127	0.0124	0.0132	0.0146
168	0.0148	0.0103	0.0078	0.0062	0.005	0.01	0.0046	0.0072
169	0.0215	0.0055	0.0083	0.007	0.0059	0.0083	0.006	0.0073
170	0.0063	0.0058	0.0073	0.0067	0.0054	0.0064	0.0039	0.0064
171	0.0427	0.0182	0.0108	0.0314	0.0228	0.0155	0.0251	0.021
172	0.0228	0.0314	0.025	0.0274	0.0231	0.0202	0.0245	0.0261
173	0.0188	0.0169	0.0167	0.021	0.0143	0.0139	0.0148	0.0144
174	0.0209	0.0128	0.012	0.0146	0.0106	0.0086	0.0109	0.0077
175	0.0124	0.0203	0.0198	0.0272	0.0244	0.0182	0.0191	0.0163
176	0.0857	0.0124	0.0282	0.0388	0.0216	0.0147	0.0224	0.019
177	0.0304	0.0283	0.0276	0.0268	0.02	0.022	0.0205	0.0202
178	0.0493	0.0199	0.0286	0.0335	0.0232	0.0195	0.0245	0.022
179	0.0229	0.0276	0.0277	0.029	0.0287	0.0233	0.0283	0.028
180	0.0113	0.0083	0.0094	0.0098	0.009	0.0079	0.0063	0.0067
181	0.0459	0.0094	0.0184	0.03	0.0126	0.0058	0.0135	0.0112
182	0.0325	0.0343	0.0212	0.0295	0.0189	0.0153	0.0206	0.0199
183	0.0406	0.0508	0.0499	0.0537	0.0459	0.0384	0.0477	0.0455
184	0.0562	0.0209	0.0274	0.041	0.0265	0.0238	0.022	0.017
185	0.0343	0.0171	0.0145	0.0271	0.0224	0.0136	0.0181	0.0149
186	0.0051	0.0069	0.0032	0.0038	0.0074	0.0049	0.0023	0.0032
187	0.0089	0.0072	0.0086	0.0071	0.0085	0.0074	0.0071	0.0075
188	0.0123	0.0038	0.0079	0.0084	0.0087	0.0053	0.0069	0.0058
189	0.0196	0.0112	0.0166	0.0191	0.0149	0.0124	0.0127	0.0109
190	0.0357	0.0178	0.0285	0.0374	0.0254	0.0218	0.0269	0.0244
191	0.0058	0.0076	0.0058	0.0072	0.0085	0.0056	0.0043	0.0049
192	0.0164	0.0119	0.0088	0.0075	0.0109	0.0099	0.011	0.0084
193	0.0102	0.0089	0.0081	0.0105	0.0088	0.0075	0.0089	0.0062
194	0.0297	0.0127	0.0311	0.0398	0.0265	0.0167	0.0281	0.0215
195	0.0122	0.008	0.0151	0.0293	0.013	0.0087	0.0089	0.0046
196	0.0059	0.0093	0.0166	0.0157	0.0135	0.0105	0.0076	0.0064
197	0.0089	0.0168	0.0114	0.0103	0.0093	0.0139	0.0091	0.0106
198	0.0256	0.0334	0.0226	0.0278	0.0261	0.0203	0.0278	0.0269

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
199	0.0341	0.0311	0.0342	0.0412	0.0321	0.0241	0.0335	0.0289
200	0.0316	0.0264	0.0228	0.0281	0.0202	0.0266	0.0196	0.0199
201	0.005	0.0058	0.0068	0.0078	0.0077	0.005	0.0033	0.003
202	0.0261	0.0222	0.0162	0.0198	0.017	0.0163	0.0173	0.0143
203	0.0503	0.051	0.0335	0.035	0.0327	0.0367	0.0337	0.0361
204	0.0738	0.0673	0.0264	0.0621	0.0574	0.0467	0.0577	0.0654
205	0.0762	0.0338	0.0454	0.0602	0.0443	0.0383	0.0483	0.0489
206	0.0173	0.0072	0.0052	0.0109	0.0086	0.0048	0.0027	0.0027
207	0.012	0.0179	0.0165	0.0194	0.0142	0.0134	0.0129	0.0126
208	0.0872	0.0489	0.0601	0.0703	0.0484	0.0411	0.0515	0.0514
209	0.0658	0.0473	0.0387	0.0501	0.0323	0.0249	0.036	0.0404
210	0.0577	0.0306	0.044	0.051	0.0357	0.0301	0.0402	0.0443
211	0.0118	0.0159	0.1482	0.0052	0.0071	0.0076	0.0068	0.0061
212	0.0088	0.007	0.0295	0.0057	0.0055	0.0089	0.0054	0.0051
213	0.007	0.0031	0.0429	0.0044	0.0033	0.0033	0.0035	0.0041
214	0.0029	0.0018	0.0279	0.0013	0.0016	0.0042	0.0018	0.0015
215	0.0013	0.0008	0.0776	0.0006	0.0011	0.0023	0.0006	0.0015
216	0.0094	0.0153	0.2181	0.0109	0.0089	0.0099	0.0093	0.0091
217	0.0072	0.0059	0.046	0.0056	0.0045	0.0049	0.0046	0.005
218	0.0076	0.0082	0.0287	0.0084	0.0075	0.0085	0.0079	0.0075
219	0.0053	0.0033	0.1058	0.0028	0.0033	0.006	0.0031	0.0034
220	0.0004	0.0007	0.1328	0.0005	0.001	0.0025	0.0005	0.001
221	0.0072	0.0121	0.1613	0.0046	0.0055	0.0092	0.005	0.007
222	0.0057	0.0028	0.1371	0.0057	0.0037	0.0033	0.0039	0.0043
223	0.0085	0.0053	0.0763	0.0066	0.0058	0.0071	0.0053	0.0044
224	0.0012	0.0029	0.0627	0.003	0.0033	0.0048	0.0032	0.0022
225	0.0013	0.0008	0.2021	0.0004	0.0011	0.0019	0.0002	0.0006
226	0.0061	0.0036	0.0925	0.0055	0.0043	0.0039	0.0045	0.0062
227	0.0033	0.0025	0.1337	0.0027	0.003	0.0037	0.0024	0.0033
228	0.0015	0.0012	0.0847	0.0012	0.0016	0.0031	0.0013	0.0015
229	0.0015	0.0008	0.0602	0.0009	0.0011	0.0033	0.0008	0.0007
230	0.0012	0.0025	0.3066	0.0027	0.0035	0.0022	0.0019	0.0026
231	0.002	0.002	0.1415	0.0033	0.0025	0.0026	0.0025	0.0039
232	0.0087	0.0029	0.0292	0.0036	0.0031	0.0034	0.0027	0.0043
233	0.006	0.0031	0.0349	0.0024	0.0024	0.0068	0.0024	0.0024
234	0.0011	0.0004	0.0706	0.0006	0.0006	0.003	0.0006	0.0003
235	0.0071	0.0042	0.3067	0.0041	0.0045	0.0036	0.0032	0.0029
236	0.0047	0.0013	0.0991	0.0023	0.0013	0.0023	0.0012	0.0014
237	0.0057	0.008	0.0273	0.0073	0.0072	0.0096	0.0073	0.0083
238	0.0052	0.0061	0.0243	0.0034	0.0041	0.0069	0.0041	0.0051

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
239	0.0075	0.0031	0.0794	0.0028	0.003	0.0063	0.0032	0.0034
240	0.0083	0.0039	0.7239	0.0047	0.0044	0.0044	0.004	0.0036
241	0.0094	0.0087	0.0983	0.0092	0.0081	0.0077	0.0082	0.007
242	0.0096	0.0086	0.0806	0.0046	0.0058	0.0116	0.0053	0.0079
243	0.0103	0.0065	0.0632	0.0067	0.0058	0.0078	0.0062	0.0072
244	0.0091	0.0107	0.0547	0.0103	0.0103	0.0127	0.0102	0.0094
245	0.003	0.0041	0.5539	0.0048	0.0044	0.0056	0.0048	0.0042
246	0.0068	0.0053	0.2842	0.0028	0.0028	0.0061	0.003	0.0038
247	0.0166	0.0155	0.0569	0.0119	0.0114	0.0146	0.0115	0.014
248	0.0131	0.0173	0.0665	0.0083	0.0105	0.0154	0.0097	0.0112
249	0.0088	0.0097	0.2077	0.0112	0.0101	0.0096	0.0111	0.0135
250	0.006	0.0045	0.8667	0.0057	0.0052	0.0035	0.0039	0.0037
251	0.0007	0.001	0.2958	0.0011	0.0014	0.003	0.0007	0.0008
252	0.0029	0.0012	0.0664	0.0016	0.0011	0.0047	0.0012	0.0014
253	0.0013	0.0004	0.1889	0.0005	0.0009	0.0029	0.0007	0.0007
254	0.002	0.0025	0.1253	0.0027	0.0028	0.0038	0.0027	0.0033
255	0.0156	0.008	0.4829	0.0067	0.0066	0.0094	0.008	0.0083
256	0.0021	0.0013	0.1192	0.0014	0.0017	0.0027	0.0009	0.0014
257	0.0029	0.004	0.087	0.0031	0.003	0.0086	0.003	0.0035
258	0.0025	0.0017	0.0206	0.0011	0.0015	0.0054	0.0014	0.0024
259	0.0097	0.0047	0.0407	0.006	0.0049	0.0077	0.0054	0.0078
260	0.0073	0.0061	0.2049	0.0094	0.0072	0.0067	0.0072	0.0083
261	0.0036	0.0043	0.0265	0.0043	0.0049	0.0021	0.0029	0.0026
262	0.0054	0.0038	0.0359	0.0032	0.0033	0.0071	0.0033	0.0024
263	0.0103	0.006	0.0129	0.0055	0.0052	0.0078	0.0051	0.0047
264	0.006	0.0084	0.0209	0.0066	0.0063	0.0077	0.0062	0.0073
265	0.0062	0.0071	0.1631	0.0045	0.0052	0.0103	0.0054	0.0076
266	0.0017	0.0029	0.3172	0.0047	0.0044	0.0037	0.0034	0.0043
267	0.0104	0.0074	0.0634	0.0078	0.007	0.0066	0.0068	0.0073
268	0.0079	0.0089	0.0173	0.0101	0.0088	0.0103	0.0095	0.0089
269	0.0205	0.021	0.0466	0.012	0.0152	0.0201	0.0142	0.0166
270	0.0068	0.0113	0.1255	0.0094	0.0093	0.0108	0.0098	0.0125
271	0.0028	0.0019	0.0442	0.0031	0.0027	0.0017	0.0017	0.0022
272	0.007	0.0105	0.1809	0.0125	0.0111	0.0113	0.0115	0.0107
273	0.0153	0.0174	0.1216	0.0094	0.0112	0.0145	0.0105	0.0098
274	0.0169	0.0239	0.156	0.0189	0.0186	0.019	0.0186	0.0189
275	0.0085	0.0116	1.1043	0.0109	0.0103	0.0112	0.0107	0.0128

Sample Size 6

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
1	0.1605	0.1484	0.041	0.066	0.0664	0.1308	0.0658	0.0463
2	0.0622	0.075	0.0664	0.0664	0.0678	0.0675	0.068	0.0604
3	0.1047	0.0909	0.0851	0.0876	0.0894	0.0878	0.0888	0.0811
4	0.1153	0.0946	0.0557	0.0475	0.0485	0.0674	0.0495	0.0487
5	0.258	0.1719	0.1672	0.2149	0.1524	0.1966	0.1474	0.1225
6	0.1071	0.0766	0.0793	0.1328	0.116	0.1297	0.1134	0.0816
7	0.0748	0.0824	0.0578	0.0605	0.0587	0.0655	0.0584	0.0642
8	0.0295	0.0281	0.0165	0.0255	0.0139	0.0088	0.0139	0.0151
9	0.0364	0.039	0.0434	0.0528	0.0395	0.0306	0.0404	0.0368
10	0.1732	0.1484	0.0857	0.1204	0.0747	0.1361	0.0719	0.0506
11	0.1638	0.0739	0.0422	0.0733	0.0658	0.0967	0.0657	0.0607
12	0.1239	0.0709	0.0461	0.0544	0.0555	0.0722	0.0542	0.0438
13	0.0807	0.0601	0.0166	0.0197	0.0141	0.0313	0.0142	0.0111
14	0.1514	0.1085	0.1429	0.1523	0.117	0.1234	0.1152	0.0886
15	0.0614	0.0364	0.0506	0.0837	0.0554	0.077	0.0533	0.0486
16	0.122	0.0758	0.0302	0.0467	0.0386	0.0679	0.0378	0.0327
17	0.0363	0.0396	0.0319	0.0284	0.0278	0.0236	0.0273	0.0228
18	0.0547	0.0711	0.0435	0.0454	0.0476	0.0409	0.0493	0.0572
19	0.082	0.1005	0.0465	0.0449	0.0458	0.057	0.0457	0.0543
20	0.2485	0.2112	0.1104	0.1871	0.1626	0.2124	0.1592	0.1367
21	0.0692	0.0655	0.0267	0.0389	0.0324	0.0412	0.0324	0.0282
22	0.1098	0.0705	0.0591	0.0577	0.0588	0.0702	0.0589	0.0446
23	0.0885	0.0995	0.0768	0.0787	0.075	0.0705	0.0739	0.0592
24	0.1031	0.0547	0.0495	0.0592	0.0583	0.0568	0.0587	0.0413
25	0.0725	0.07	0.0379	0.0507	0.0406	0.0698	0.0366	0.0318
26	0.2492	0.1412	0.08	0.1072	0.0917	0.148	0.0916	0.0755
27	0.097	0.0491	0.0292	0.0311	0.0301	0.049	0.028	0.0268
28	0.0603	0.0542	0.0593	0.0654	0.0506	0.0417	0.0486	0.0433
29	0.0542	0.0519	0.0431	0.0468	0.0455	0.0505	0.0412	0.0381
30	0.0305	0.0276	0.0453	0.0654	0.0488	0.0473	0.0454	0.0369
31	0.0517	0.0361	0.0376	0.0658	0.0534	0.0687	0.0499	0.035
32	0.2008	0.0864	0.0556	0.0587	0.0709	0.0937	0.0687	0.0566
33	0.1028	0.0676	0.0631	0.0735	0.074	0.0767	0.0685	0.0467
34	0.0355	0.0257	0.0533	0.0663	0.0435	0.0359	0.0378	0.0345
35	0.0879	0.05	0.065	0.0958	0.0708	0.0855	0.0718	0.0596
36	0.1397	0.1142	0.0963	0.1261	0.0979	0.1337	0.0914	0.0722
37	0.0882	0.1062	0.0755	0.0874	0.0758	0.1008	0.0739	0.0602
38	0.0731	0.0939	0.0572	0.0636	0.0592	0.0692	0.0602	0.0577

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
39	0.0851	0.0526	0.0237	0.0227	0.0229	0.0368	0.0223	0.0233
40	0.0784	0.0426	0.0662	0.0866	0.0676	0.0558	0.0614	0.05
41	0.1075	0.0914	0.0786	0.1083	0.0916	0.1217	0.0878	0.0725
42	0.0378	0.0304	0.0191	0.0258	0.0239	0.024	0.0255	0.0178
43	0.0332	0.0455	0.0323	0.0367	0.0338	0.0283	0.0346	0.0307
44	0.0358	0.0437	0.038	0.0374	0.0401	0.0358	0.0389	0.0443
45	0.0516	0.0487	0.053	0.0658	0.0537	0.0553	0.0465	0.0397
46	0.1347	0.0728	0.1103	0.1624	0.1217	0.128	0.1196	0.1046
47	0.065	0.0827	0.0445	0.0409	0.0414	0.0469	0.0404	0.0388
48	0.0426	0.0602	0.0756	0.0904	0.068	0.0536	0.0673	0.0644
49	0.0757	0.0471	0.0652	0.0739	0.0558	0.0437	0.0516	0.0437
50	0.1171	0.0539	0.0995	0.1186	0.086	0.0746	0.0833	0.0697
51	0.0188	0.0357	0.0631	0.091	0.0592	0.0489	0.0566	0.0394
52	0.1186	0.0845	0.0684	0.0848	0.0766	0.0994	0.0746	0.0733
53	0.2085	0.1293	0.1533	0.184	0.1411	0.1615	0.1349	0.0988
54	0.0631	0.0615	0.0734	0.0785	0.057	0.0705	0.0535	0.0406
55	0.1466	0.0866	0.0733	0.1102	0.0848	0.111	0.0842	0.0595
56	0.071	0.0397	0.046	0.0566	0.0463	0.0577	0.0445	0.042
57	0.0294	0.0397	0.0364	0.0387	0.0373	0.0348	0.0359	0.0378
58	0.0524	0.0533	0.056	0.0658	0.0493	0.0577	0.0454	0.0326
59	0.1172	0.0855	0.0742	0.0747	0.0656	0.0717	0.0649	0.0567
60	0.0903	0.0354	0.0405	0.0588	0.0434	0.0471	0.0423	0.0393
61	0.0292	0.0303	0.037	0.0474	0.036	0.0436	0.031	0.0234
62	0.024	0.0317	0.0215	0.0236	0.0235	0.0244	0.0195	0.0137
63	0.0403	0.0248	0.0341	0.0448	0.028	0.0257	0.0284	0.0235
64	0.063	0.0771	0.034	0.0392	0.0321	0.0492	0.0336	0.0394
65	0.0996	0.0694	0.0679	0.0915	0.0611	0.0825	0.0574	0.0516
66	0.0696	0.0663	0.0272	0.0465	0.0454	0.0707	0.0434	0.0374
67	0.0715	0.0525	0.0478	0.0548	0.0507	0.0422	0.05	0.0468
68	0.0637	0.0685	0.0687	0.0684	0.0843	0.0698	0.0836	0.0729
69	0.0988	0.0528	0.0911	0.0924	0.0741	0.0714	0.0728	0.0607
70	0.2814	0.1557	0.2171	0.2499	0.2027	0.1794	0.2121	0.1653
71	0.1405	0.0631	0.0415	0.0692	0.0614	0.0942	0.0568	0.0417
72	0.0259	0.0243	0.0228	0.0224	0.0182	0.0155	0.0186	0.023
73	0.0569	0.0396	0.0236	0.0286	0.0181	0.021	0.0175	0.0197
74	0.0913	0.0711	0.0512	0.0597	0.0478	0.0621	0.0479	0.0337
75	0.1981	0.1792	0.1663	0.1998	0.1362	0.179	0.1307	0.1072
76	0.1535	0.0816	0.068	0.1256	0.0973	0.1383	0.0898	0.0685
77	0.1192	0.0718	0.058	0.0711	0.06	0.0841	0.0568	0.0524
78	0.1486	0.0784	0.1517	0.1734	0.1268	0.1003	0.1295	0.1049

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
79	0.1467	0.088	0.1177	0.1204	0.1069	0.1073	0.1044	0.088
80	0.0773	0.0925	0.0904	0.1305	0.116	0.1144	0.1183	0.1086
81	0.0442	0.0192	0.0458	0.0686	0.0373	0.0238	0.0364	0.034
82	0.068	0.0452	0.06	0.0788	0.0429	0.0404	0.0399	0.037
83	0.0249	0.0169	0.0171	0.0187	0.0191	0.0114	0.0199	0.0221
84	0.0525	0.05	0.0307	0.0334	0.0394	0.0339	0.0405	0.0386
85	0.0176	0.017	0.0131	0.0211	0.0282	0.0232	0.0197	0.0134
86	0.0565	0.0676	0.0595	0.0765	0.0458	0.037	0.0443	0.0411
87	0.0339	0.0367	0.0267	0.0369	0.0213	0.0167	0.0198	0.0137
88	0.0385	0.0187	0.0312	0.0386	0.0299	0.0204	0.0315	0.0252
89	0.04	0.0208	0.0277	0.0287	0.0208	0.0227	0.0221	0.0205
90	0.0271	0.0232	0.0273	0.0337	0.026	0.03	0.0221	0.0209
91	0.0985	0.0841	0.0668	0.0791	0.0598	0.0685	0.0605	0.0584
92	0.069	0.0398	0.0362	0.0452	0.0303	0.05	0.0289	0.023
93	0.0898	0.0452	0.025	0.0372	0.0283	0.0357	0.0244	0.0196
94	0.045	0.027	0.0171	0.0217	0.0172	0.0221	0.0151	0.0154
95	0.0194	0.0255	0.0224	0.0418	0.0382	0.0361	0.0258	0.021
96	0.0511	0.0289	0.0554	0.0809	0.0569	0.0577	0.053	0.0499
97	0.0379	0.0403	0.0342	0.0365	0.027	0.0334	0.024	0.0269
98	0.0368	0.0295	0.0252	0.027	0.0266	0.0284	0.0229	0.0235
99	0.025	0.0168	0.0131	0.0111	0.0069	0.0098	0.0067	0.0088
100	0.0286	0.0175	0.0175	0.0265	0.0225	0.0201	0.0116	0.0076
101	0.0214	0.0289	0.0538	0.0601	0.0431	0.0387	0.0401	0.034
102	0.0103	0.0289	0.0278	0.0359	0.0277	0.027	0.0249	0.022
103	0.0516	0.0273	0.0168	0.0178	0.0174	0.0132	0.0171	0.012
104	0.047	0.0337	0.0399	0.0423	0.0393	0.0331	0.0397	0.0353
105	0.1037	0.0467	0.0514	0.0572	0.0487	0.0528	0.0442	0.0416
106	0.0545	0.0323	0.0653	0.0857	0.0549	0.0531	0.0517	0.0425
107	0.0438	0.038	0.0307	0.0388	0.0299	0.0233	0.0274	0.025
108	0.0499	0.0624	0.0574	0.0616	0.0505	0.0418	0.0504	0.0508
109	0.0169	0.027	0.0251	0.022	0.0255	0.028	0.0256	0.0271
110	0.0732	0.0695	0.0486	0.0572	0.0456	0.0704	0.0411	0.0435
111	0.1072	0.0892	0.1192	0.1734	0.1234	0.1225	0.1162	0.1058
112	0.0445	0.0416	0.0117	0.016	0.0117	0.0129	0.0126	0.0155
113	0.0697	0.0851	0.0498	0.0587	0.0448	0.0458	0.0435	0.0415
114	0.1173	0.1127	0.0742	0.0878	0.0813	0.0733	0.0798	0.0725
115	0.0374	0.0225	0.0409	0.0507	0.038	0.0337	0.0302	0.0245
116	0.0369	0.0091	0.049	0.0885	0.0347	0.0367	0.0284	0.0183
117	0.0573	0.0424	0.0853	0.1009	0.0729	0.0531	0.0709	0.0612
118	0.1369	0.1115	0.1269	0.1404	0.1037	0.107	0.0997	0.0957

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
119	0.0975	0.0777	0.1638	0.1793	0.1268	0.1124	0.1181	0.1057
120	0.0888	0.0425	0.0659	0.0754	0.0516	0.0507	0.0497	0.0482
121	0.0313	0.0257	0.0328	0.0441	0.0341	0.0327	0.0234	0.0169
122	0.0337	0.0182	0.01	0.0109	0.0198	0.0166	0.0151	0.011
123	0.0567	0.0201	0.0295	0.028	0.03	0.0225	0.031	0.0258
124	0.0569	0.0582	0.0445	0.0428	0.0509	0.0556	0.0506	0.0425
125	0.1158	0.0736	0.0939	0.1102	0.0849	0.0739	0.09	0.0875
126	0.0124	0.0189	0.0095	0.0144	0.0177	0.0226	0.0124	0.0086
127	0.0353	0.0324	0.0169	0.0166	0.0216	0.021	0.0217	0.0206
128	0.0271	0.0408	0.0194	0.0154	0.0136	0.0214	0.013	0.0184
129	0.0334	0.032	0.0351	0.0404	0.026	0.0274	0.0261	0.025
130	0.058	0.0553	0.0487	0.0641	0.0384	0.0508	0.0329	0.0295
131	0.034	0.0257	0.021	0.0254	0.0289	0.0311	0.0238	0.0235
132	0.0715	0.0498	0.0334	0.037	0.0407	0.0414	0.0398	0.0371
133	0.0655	0.0552	0.0426	0.0552	0.044	0.0386	0.0438	0.0437
134	0.0477	0.0199	0.0268	0.0363	0.0199	0.0127	0.0204	0.017
135	0.0654	0.0396	0.0558	0.0731	0.0455	0.0399	0.0428	0.0377
136	0.0355	0.0258	0.0243	0.0251	0.019	0.0249	0.017	0.0176
137	0.044	0.0619	0.0237	0.0276	0.0247	0.0276	0.024	0.0221
138	0.0662	0.1325	0.0568	0.057	0.0474	0.0478	0.0463	0.0485
139	0.0954	0.0676	0.0866	0.1042	0.066	0.068	0.0644	0.0583
140	0.118	0.0524	0.1308	0.1399	0.0994	0.0786	0.101	0.0928
141	0.0228	0.0359	0.0418	0.0604	0.0436	0.0476	0.0343	0.0301
142	0.0399	0.0606	0.0514	0.0687	0.0535	0.0599	0.0463	0.0454
143	0.1499	0.1136	0.1843	0.2017	0.1594	0.1386	0.1549	0.1376
144	0.2337	0.1191	0.1613	0.1901	0.1369	0.1324	0.1352	0.1216
145	0.1138	0.0589	0.0513	0.0636	0.0423	0.0493	0.0442	0.0422
146	0.0475	0.0742	0.0265	0.0352	0.0222	0.0207	0.0232	0.0223
147	0.0382	0.0135	0.0209	0.028	0.0147	0.0111	0.016	0.0115
148	0.0116	0.0292	0.0135	0.0136	0.0161	0.0189	0.0156	0.0172
149	0.01	0.0068	0.003	0.0035	0.004	0.0062	0.0041	0.0041
150	0.0069	0.0039	0.0065	0.0036	0.0044	0.0045	0.0029	0.0035
151	0.0797	0.0392	0.0474	0.065	0.0364	0.03	0.0384	0.0369
152	0.0353	0.0303	0.0271	0.0268	0.0213	0.0187	0.0218	0.019
153	0.0156	0.0196	0.0203	0.0206	0.0146	0.0161	0.0149	0.0161
154	0.0171	0.0093	0.0126	0.011	0.008	0.0136	0.0082	0.0129
155	0.0066	0.0056	0.0079	0.0025	0.0047	0.0061	0.0025	0.006
156	0.0285	0.0382	0.0374	0.0557	0.0334	0.0197	0.0361	0.0321
157	0.0414	0.0348	0.059	0.048	0.0384	0.0281	0.0409	0.0441
158	0.0565	0.0499	0.0419	0.0494	0.0427	0.0446	0.0433	0.041

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
159	0.0234	0.0191	0.0258	0.0187	0.0194	0.02	0.0175	0.0209
160	0.0063	0.0059	0.0043	0.0023	0.0062	0.0038	0.0021	0.0035
161	0.0207	0.017	0.0317	0.0462	0.0312	0.0225	0.0291	0.0258
162	0.0085	0.0056	0.0101	0.0071	0.0059	0.0063	0.0061	0.0084
163	0.0195	0.0113	0.0115	0.0112	0.0132	0.0113	0.011	0.0126
164	0.0132	0.0083	0.0076	0.0055	0.0081	0.009	0.0065	0.0087
165	0.0381	0.0296	0.0204	0.0341	0.0307	0.0252	0.023	0.016
166	0.0356	0.0198	0.028	0.0348	0.0231	0.0218	0.0213	0.0196
167	0.0256	0.0224	0.029	0.0224	0.0187	0.0231	0.019	0.0235
168	0.0202	0.0117	0.0134	0.0146	0.0161	0.0135	0.017	0.013
169	0.0104	0.0107	0.0081	0.0056	0.0053	0.0103	0.0062	0.0086
170	0.0273	0.0195	0.0333	0.0189	0.0167	0.0186	0.0106	0.0117
171	0.0482	0.0171	0.013	0.0168	0.0091	0.0139	0.0111	0.0127
172	0.0272	0.0318	0.0355	0.0469	0.0328	0.027	0.0345	0.0287
173	0.0121	0.015	0.0168	0.0209	0.0145	0.0128	0.0152	0.0128
174	0.0377	0.0077	0.0236	0.0136	0.0092	0.0086	0.0087	0.0089
175	0.0088	0.0141	0.0196	0.0242	0.0195	0.0146	0.0163	0.0144
176	0.0381	0.0256	0.0409	0.0541	0.0355	0.0289	0.0364	0.0345
177	0.073	0.0505	0.0656	0.0795	0.0598	0.0449	0.0626	0.0539
178	0.1113	0.0905	0.076	0.0843	0.0745	0.0699	0.079	0.0825
179	0.0252	0.0382	0.0255	0.0339	0.0349	0.03	0.0333	0.0341
180	0.03	0.0289	0.0224	0.0493	0.0447	0.0271	0.0388	0.0298
181	0.0668	0.0279	0.0743	0.0637	0.0412	0.0366	0.0409	0.0383
182	0.0323	0.0439	0.0453	0.0588	0.035	0.0275	0.0374	0.0369
183	0.1291	0.0844	0.0791	0.1266	0.1052	0.0836	0.1057	0.0838
184	0.0683	0.0364	0.0263	0.0673	0.0487	0.0401	0.0478	0.038
185	0.0156	0.0228	0.0554	0.0424	0.0354	0.0254	0.0247	0.0215
186	0.0055	0.0052	0.0069	0.006	0.0096	0.0045	0.0032	0.0046
187	0.0081	0.0062	0.0055	0.0038	0.0054	0.0068	0.0047	0.0076
188	0.0139	0.0105	0.0172	0.0126	0.0113	0.0122	0.0099	0.0108
189	0.0138	0.012	0.0347	0.0165	0.0138	0.0154	0.012	0.012
190	0.0572	0.0271	0.0739	0.0419	0.0321	0.0278	0.0364	0.0308
191	0.0069	0.006	0.0065	0.007	0.009	0.0043	0.0045	0.0031
192	0.0088	0.0143	0.0088	0.0079	0.014	0.0112	0.0122	0.0124
193	0.0138	0.0183	0.0063	0.0095	0.0106	0.0132	0.0095	0.0137
194	0.0393	0.022	0.0327	0.0382	0.0276	0.0229	0.0274	0.0304
195	0.0972	0.0528	0.0251	0.0745	0.0596	0.0459	0.0611	0.0544
196	0.0162	0.0184	0.023	0.032	0.0318	0.0199	0.0208	0.0136
197	0.033	0.0495	0.0354	0.034	0.0347	0.0329	0.0348	0.0319
198	0.0262	0.025	0.0154	0.0178	0.0163	0.0157	0.0174	0.0138

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
199	0.0373	0.0455	0.0306	0.0352	0.0296	0.0334	0.0294	0.0264
200	0.0809	0.0397	0.0559	0.0762	0.0503	0.0509	0.0492	0.0424
201	0.009	0.0163	0.0155	0.0193	0.0188	0.0137	0.0129	0.0178
202	0.0541	0.0439	0.0331	0.0348	0.035	0.0369	0.0333	0.0389
203	0.0573	0.0293	0.039	0.042	0.0327	0.0317	0.0334	0.0299
204	0.1002	0.1081	0.0738	0.0952	0.0741	0.0651	0.0791	0.0732
205	0.1075	0.064	0.0745	0.08	0.066	0.0617	0.0727	0.0626
206	0.012	0.0208	0.0338	0.03	0.0266	0.0235	0.0152	0.0192
207	0.0486	0.0297	0.0505	0.0652	0.0427	0.0342	0.0423	0.0384
208	0.0787	0.0606	0.0663	0.0811	0.0596	0.0497	0.0634	0.0816
209	0.118	0.0692	0.0584	0.0714	0.0551	0.0498	0.0602	0.0736
210	0.0926	0.0767	0.0487	0.081	0.0705	0.0631	0.097	0.1116
211	0.0159	0.0192	3.2502	0.013	0.0134	0.0145	0.013	0.0144
212	0.0032	0.0059	0.2414	0.0079	0.0058	0.0048	0.0063	0.0071
213	0.0075	0.0059	0.1148	0.0042	0.0046	0.0073	0.0045	0.0049
214	0.0041	0.0049	0.1255	0.004	0.0046	0.0091	0.0038	0.0037
215	0.0007	0.0006	0.2908	0.0004	0.0009	0.0027	0.0004	0.0026
216	0.0225	0.0235	1.6273	0.0159	0.0176	0.0181	0.0169	0.0205
217	0.0038	0.0055	0.0482	0.0069	0.0052	0.0063	0.0057	0.0071
218	0.0116	0.0104	0.2546	0.0106	0.0103	0.0128	0.0101	0.0097
219	0.0048	0.0034	0.0861	0.0034	0.0037	0.0071	0.0036	0.0046
220	0.0015	0.0017	6.2056	0.0014	0.0023	0.0029	0.0014	0.0038
221	0.0187	0.0194	0.1999	0.0167	0.0152	0.0165	0.0156	0.0239
222	0.0077	0.0065	0.0751	0.0081	0.0066	0.0078	0.0073	0.013
223	0.0096	0.0129	1.3283	0.0137	0.0129	0.0134	0.0142	0.0182
224	0.004	0.0043	1.1455	0.0042	0.0044	0.0039	0.0039	0.0063
225	0.0015	0.002	2.2774	0.0011	0.0027	0.0013	0.0011	0.0039
226	0.0052	0.0039	0.974	0.0075	0.0058	0.0033	0.0064	0.0111
227	0.0032	0.003	0.6174	0.003	0.0033	0.0039	0.0029	0.0064
228	0.0017	0.0032	0.3031	0.0028	0.0035	0.0047	0.0029	0.0094
229	0.0011	0.0011	0.2616	0.0012	0.0013	0.0044	0.0013	0.0034
230	0.0023	0.0039	0.8206	0.0041	0.005	0.0027	0.0058	0.0086
231	0.0057	0.0076	0.2724	0.0078	0.0071	0.0073	0.0089	0.0112
232	0.0069	0.0051	0.2793	0.0047	0.0047	0.0066	0.0045	0.0025
233	0.0069	0.0054	0.0643	0.0059	0.0058	0.0052	0.0058	0.0086
234	0.0015	0.0033	0.0414	0.0039	0.0036	0.0046	0.0032	0.0034
235	0.0107	0.0069	0.3074	0.007	0.0078	0.005	0.0055	0.0092
236	0.0109	0.0093	0.1019	0.0086	0.0082	0.0104	0.0091	0.0075
237	0.0074	0.008	0.1799	0.0089	0.008	0.0091	0.0078	0.0092
238	0.0057	0.0057	0.0695	0.0066	0.0051	0.0087	0.0053	0.0063

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
239	0.0046	0.004	0.2096	0.0025	0.0022	0.0071	0.0025	0.0109
240	0.0054	0.0058	2.158	0.0072	0.0063	0.0073	0.006	0.0074
241	0.0066	0.0041	0.3653	0.0045	0.0034	0.0062	0.0035	0.0057
242	0.0079	0.005	0.0442	0.0063	0.0045	0.0065	0.005	0.0046
243	0.0177	0.0165	0.344	0.0168	0.0157	0.0143	0.0156	0.0054
244	0.0156	0.017	0.2688	0.0164	0.0159	0.0166	0.0165	0.0066
245	0.0095	0.0067	1.5362	0.0081	0.0079	0.0049	0.0063	0.0079
246	0.0126	0.0095	0.3862	0.01	0.0089	0.0114	0.0094	0.0177
247	0.0065	0.0026	0.2506	0.0063	0.003	0.0037	0.0035	0.0156
248	0.0541	0.0396	1.3309	0.0434	0.0407	0.0399	0.0412	0.0095
249	0.0161	0.0175	0.4743	0.0132	0.015	0.0164	0.0163	0.0126
250	0.0076	0.0098	0.2487	0.0125	0.0113	0.0094	0.0112	0.0065
251	0.0013	0.0023	0.7526	0.0021	0.0031	0.0024	0.0012	0.006
252	0.0023	0.0023	0.1643	0.0017	0.0026	0.0048	0.0018	0.0043
253	0.0055	0.0054	0.1985	0.0048	0.0053	0.0089	0.0053	0.008
254	0.0067	0.0028	0.8062	0.0035	0.0031	0.0043	0.0031	0.0009
255	0.0128	0.0083	0.2209	0.0117	0.0096	0.0092	0.0093	0.0075
256	0.0016	0.0017	0.3535	0.0024	0.0033	0.0024	0.0013	0.0024
257	0.0053	0.0043	0.0749	0.0044	0.0046	0.0081	0.0041	0.0087
258	0.0103	0.0091	0.0392	0.0069	0.0081	0.0106	0.0077	0.0096
259	0.0124	0.0054	0.1258	0.0049	0.0042	0.0071	0.0046	0.009
260	0.0329	0.0198	0.3226	0.0224	0.0193	0.0204	0.0235	0.0294
261	0.0035	0.0021	0.4113	0.0025	0.0025	0.0029	0.002	0.0037
262	0.0115	0.0131	1.3044	0.0122	0.0117	0.0152	0.0118	0.0152
263	0.0147	0.0169	0.1764	0.0164	0.0165	0.0181	0.0162	0.0148
264	0.0159	0.013	0.2591	0.013	0.0124	0.014	0.0125	0.0116
265	0.0207	0.0201	1.7903	0.0234	0.0213	0.0169	0.0231	0.0211
266	0.0039	0.0033	1.1306	0.0038	0.004	0.0028	0.003	0.0036
267	0.0115	0.0099	0.0828	0.0067	0.0077	0.0145	0.0078	0.0103
268	0.0148	0.0132	0.0476	0.0127	0.012	0.0133	0.013	0.0202
269	0.0214	0.0227	0.1095	0.0176	0.0193	0.0209	0.019	0.0237
270	0.0306	0.0239	0.6243	0.0197	0.0211	0.0263	0.0227	0.0198
271	0.0077	0.0037	0.2349	0.0047	0.0043	0.0028	0.0034	0.0056
272	0.0074	0.0076	0.3033	0.0043	0.0045	0.0068	0.0045	0.0055
273	0.0307	0.0362	0.6279	0.0354	0.0344	0.0281	0.0362	0.0333
274	0.0427	0.0282	1.0412	0.0249	0.0252	0.0249	0.0249	0.0308
275	0.0303	0.0185	8.2988	0.0158	0.0167	0.0168	0.0163	0.0259

Sample Size 3

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
1	0.1259	0.0555	0.1242	0.1937	0.1568	0.0893	0.1537	0.1278
2	0.1866	0.1739	0.1476	0.157	0.1391	0.1354	0.1302	0.1132
3	0.1343	0.0972	0.086	0.11	0.1037	0.0847	0.0965	0.0828
4	0.0705	0.081	0.059	0.0657	0.0647	0.0558	0.0698	0.0639
5	0.374	0.1618	0.1749	0.2271	0.167	0.159	0.1699	0.1098
6	0.2407	0.1314	0.0965	0.1037	0.1124	0.113	0.1118	0.0905
7	0.127	0.2073	0.0866	0.1052	0.1085	0.1435	0.1059	0.1073
8	0.1085	0.0794	0.1213	0.1289	0.1087	0.0905	0.1116	0.1122
9	0.0956	0.0718	0.0414	0.0435	0.0341	0.0325	0.0343	0.0353
10	0.3159	0.1383	0.2049	0.2334	0.1821	0.1671	0.1814	0.1577
11	0.3397	0.2626	0.2475	0.2675	0.242	0.2445	0.2413	0.238
12	0.3555	0.2016	0.1135	0.2306	0.2049	0.2019	0.1932	0.1175
13	0.1377	0.0383	0.0925	0.0867	0.0775	0.0485	0.0787	0.0743
14	0.3567	0.124	0.1971	0.221	0.1848	0.1669	0.1768	0.1602
15	0.3103	0.2872	0.3938	0.5872	0.4894	0.38	0.4904	0.4099
16	0.2531	0.1203	0.2419	0.3172	0.2723	0.2045	0.2782	0.2166
17	0.2179	0.1326	0.0821	0.0942	0.0892	0.0941	0.0895	0.0892
18	0.1228	0.0904	0.0929	0.0965	0.1162	0.0807	0.1079	0.103
19	0.1588	0.163	0.0904	0.091	0.1182	0.1141	0.1137	0.1478
20	0.0767	0.0829	0.2029	0.0901	0.0832	0.0791	0.0882	0.0971
21	0.5614	0.1258	0.1444	0.2209	0.1943	0.1932	0.1932	0.1197
22	0.0604	0.0407	0.1366	0.1715	0.1687	0.1059	0.1577	0.1093
23	0.4211	0.2662	0.2033	0.2703	0.26	0.2371	0.2456	0.2453
24	0.0813	0.1105	0.0648	0.0686	0.0676	0.0685	0.0712	0.0798
25	0.2168	0.1501	0.185	0.3562	0.2942	0.2348	0.2744	0.1865
26	0.4255	0.2835	0.1632	0.2673	0.2027	0.2484	0.1973	0.1374
27	0.198	0.1413	0.0973	0.1044	0.0994	0.1035	0.1011	0.1099
28	0.282	0.1402	0.1711	0.2018	0.1599	0.1486	0.1572	0.1442
29	0.0609	0.0782	0.1246	0.1319	0.1054	0.0829	0.0987	0.096
30	0.2176	0.1918	0.1281	0.1846	0.1519	0.1834	0.1714	0.1398
31	0.1302	0.0801	0.1052	0.1443	0.1166	0.1047	0.1178	0.0936
32	0.1093	0.0387	0.0535	0.0962	0.0793	0.0587	0.1676	0.1637
33	0.1384	0.1074	0.0834	0.0725	0.0889	0.0924	0.0872	0.0944
34	0.0765	0.0932	0.0757	0.1036	0.0709	0.074	0.0658	0.0508
35	0.2452	0.2141	0.148	0.3129	0.257	0.2171	0.2425	0.2288
36	0.1206	0.0835	0.1131	0.1481	0.1051	0.0857	0.1031	0.09
37	0.0591	0.0702	0.0736	0.09	0.0813	0.0603	0.0764	0.0581
38	0.0926	0.06	0.0709	0.066	0.0575	0.0562	0.057	0.0629

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
39	0.174	0.1274	0.1235	0.1383	0.1733	0.1271	0.162	0.1416
40	0.2614	0.2568	0.149	0.3122	0.2695	0.2415	1.2091	1.1352
41	0.242	0.096	0.1325	0.2068	0.1571	0.1424	0.1483	0.0858
42	0.1563	0.1577	0.076	0.086	0.1005	0.0942	0.1013	0.1099
43	0.093	0.064	0.0667	0.0684	0.0675	0.0599	0.0657	0.0781
44	0.1345	0.1101	0.0812	0.123	0.1075	0.0907	0.1037	0.0995
45	0.087	0.026	0.0623	0.0971	0.0622	0.0456	0.0647	0.0522
46	0.164	0.052	0.041	0.0953	0.0558	0.0642	0.0498	0.0233
47	0.3591	0.1244	0.0891	0.12	0.1021	0.1128	0.1049	0.0805
48	0.1851	0.1999	0.1654	0.1917	0.1893	0.1803	0.1913	0.1844
49	0.1952	0.1349	0.1458	0.1445	0.1364	0.1281	0.1243	0.1066
50	0.1247	0.0346	0.0699	0.0931	0.0764	0.0564	0.0794	0.0696
51	0.17	0.1102	0.12	0.1554	0.1502	0.1246	0.1495	0.1319
52	0.3746	0.1906	0.2445	0.3501	0.2613	0.2538	0.256	0.1779
53	0.3441	0.3042	0.2987	0.3631	0.2919	0.2902	0.2626	0.1561
54	0.0841	0.081	0.1134	0.1298	0.0911	0.0783	0.0878	0.0692
55	0.1161	0.0684	0.045	0.0727	0.054	0.0465	0.0578	0.0543
56	0.5603	0.1308	0.2111	0.2896	0.257	0.2373	0.2518	0.1827
57	0.2487	0.1979	0.0924	0.164	0.1399	0.1919	0.1495	0.1044
58	0.2852	0.153	0.2919	0.3547	0.2552	0.2121	0.2388	0.1582
59	0.1907	0.0543	0.139	0.1642	0.1389	0.097	0.1307	0.088
60	0.1684	0.0893	0.1398	0.1766	0.1236	0.108	0.1091	0.0941
61	0.1718	0.1159	0.0896	0.1149	0.1008	0.1062	0.09	0.067
62	0.1324	0.1434	0.0836	0.0796	0.0764	0.1027	0.0788	0.0879
63	0.0634	0.0862	0.0434	0.0513	0.0725	0.0584	0.0633	0.0737
64	0.1136	0.054	0.1137	0.1165	0.1109	0.0613	0.1128	0.1329
65	0.2533	0.1946	0.15	0.3043	0.2482	0.2028	0.2599	0.2326
66	0.2947	0.2132	0.6617	0.2326	0.2053	0.2383	0.1938	0.1242
67	0.1293	0.0943	0.1177	0.1241	0.1184	0.0818	0.1231	0.1202
68	0.0881	0.0443	0.0462	0.0569	0.0507	0.0427	0.0481	0.048
69	0.1357	0.1567	0.1435	0.1418	0.1369	0.1113	0.1247	0.1293
70	0.4187	0.2297	0.2763	0.3541	0.2644	0.2732	0.2369	0.1823
71	0.1713	0.2228	0.167	0.2401	0.1832	0.2216	0.168	0.138
72	0.2116	0.0644	0.071	0.1047	0.1054	0.0682	0.0954	0.073
73	0.1678	0.0843	0.0392	0.0447	0.0495	0.0532	0.051	0.0609
74	0.1347	0.0334	0.1185	0.1357	0.1067	0.0683	0.0996	0.0715
75	0.1854	0.1085	0.7759	0.2598	0.1889	0.142	0.1928	0.1578
76	0.1277	0.1163	0.8165	0.1048	0.0995	0.0978	0.1039	0.1302
77	0.3336	0.2215	0.0555	0.1801	0.1659	0.1794	0.167	0.1317
78	0.1301	0.1463	0.0671	0.0863	0.0772	0.1007	0.0771	0.0704

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
79	0.1626	0.0786	0.1609	0.2174	0.1438	0.0948	0.1309	0.0809
80	0.172	0.1183	0.1109	0.1218	0.1076	0.1051	0.1039	0.1056
81	0.2478	0.1312	0.0762	0.1051	0.0665	0.086	0.0657	0.0528
82	0.1991	0.1281	0.1158	0.1388	0.1093	0.0837	0.1092	0.0801
83	0.0787	0.0613	0.0509	0.0558	0.0584	0.0499	0.0566	0.0718
84	0.0417	0.0813	0.0792	0.0682	0.0701	0.0618	0.0515	0.0561
85	0.0581	0.0386	0.0591	0.0595	0.0473	0.0447	0.0457	0.0452
86	0.1627	0.1273	0.098	0.1224	0.083	0.1038	0.0814	0.0688
87	0.1452	0.1548	0.1078	0.1344	0.1036	0.0997	0.0966	0.0848
88	0.15	0.1089	0.0339	0.1181	0.1092	0.0791	0.1026	0.1094
89	0.1176	0.1376	0.1088	0.1189	0.1227	0.1156	0.1125	0.1196
90	0.0403	0.0433	0.097	0.1291	0.0915	0.0682	0.0697	0.0471
91	0.3975	0.0702	0.1809	0.2376	0.1687	0.1273	0.1648	0.1539
92	0.0723	0.0806	0.0738	0.097	0.073	0.078	0.0623	0.0678
93	0.1026	0.1124	0.1813	0.2465	0.1914	0.1427	0.1758	0.1401
94	0.0708	0.03	0.0638	0.0645	0.0529	0.034	0.0448	0.0433
95	0.0773	0.0574	0.0645	0.0559	0.0572	0.0542	0.0537	0.0406
96	0.0918	0.0589	0.1159	0.1597	0.1197	0.0861	0.1138	0.0939
97	0.0888	0.0589	0.0429	0.0464	0.0534	0.0534	0.0455	0.046
98	0.1298	0.0926	0.0475	0.0627	0.0636	0.0673	0.0689	0.0674
99	0.0821	0.0955	0.0781	0.0702	0.079	0.082	0.0755	0.0414
100	0.0304	0.0104	0.066	0.0609	0.0415	0.0225	0.0322	0.0179
101	0.0769	0.0616	0.0514	0.0555	0.0533	0.0573	0.0565	0.0627
102	0.0887	0.0846	0.0776	0.096	0.1041	0.0772	0.0919	0.1209
103	0.0858	0.0629	0.029	0.0269	0.0302	0.0291	0.0227	0.0215
104	0.0585	0.0778	0.0349	0.058	0.0582	0.0476	0.055	0.0558
105	0.0424	0.0439	0.0374	0.0689	0.0526	0.0451	0.0631	0.0599
106	0.0646	0.0376	0.0482	0.095	0.0724	0.0547	0.0681	0.0409
107	0.1358	0.0834	0.055	0.0587	0.0448	0.0502	0.0471	0.0483
108	0.1547	0.1327	0.0404	0.0605	0.0615	0.0701	0.0598	0.0606
109	0.1204	0.0916	0.0597	0.0661	0.061	0.0554	0.0579	0.0737
110	0.263	0.1089	0.1749	0.1481	0.1279	0.1194	0.1291	0.1112
111	0.1305	0.0794	0.2314	0.3016	0.2024	0.1475	0.1851	0.1317
112	0.1994	0.117	0.1765	0.2071	0.1551	0.1203	0.1514	0.124
113	0.1029	0.0528	0.0551	0.0527	0.0517	0.0466	0.0507	0.0542
114	0.0691	0.0912	0.126	0.1522	0.102	0.083	0.0802	0.0477
115	0.0503	0.0538	0.0964	0.0523	0.0378	0.0424	0.0334	0.0307
116	0.0506	0.0559	0.9542	0.0543	0.0419	0.0318	0.0425	0.057
117	0.2458	0.2293	0.2492	0.31	0.2405	0.2295	0.2411	0.1777
118	0.1417	0.1244	0.1969	0.1774	0.1296	0.1198	0.1304	0.1042

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
119	0.1408	0.1137	0.0833	0.0926	0.0841	0.0919	0.082	0.091
120	0.1951	0.1798	0.117	0.1561	0.1414	0.1522	0.1356	0.1033
121	0.1526	0.0602	0.0807	0.0873	0.0644	0.071	0.059	0.0587
122	0.0864	0.1116	0.0768	0.0914	0.0882	0.1036	0.0892	0.0938
123	0.0832	0.0714	0.071	0.0711	0.0637	0.067	0.0657	0.0639
124	0.0526	0.054	0.0692	0.0338	0.0288	0.0415	0.0264	0.0332
125	0.1344	0.1072	0.097	0.124	0.0931	0.112	0.3437	0.242
126	0.1419	0.032	0.363	0.0707	0.0666	0.0517	0.0547	0.0322
127	0.0736	0.0728	0.1159	0.0913	0.1012	0.0883	0.0904	0.0896
128	0.1181	0.1486	0.0615	0.0723	0.0635	0.0793	0.057	0.0667
129	0.0626	0.0609	0.4111	0.0354	0.0388	0.0407	0.0368	0.032
130	0.4161	0.2007	0.1449	0.2644	0.2114	0.1945	0.2075	0.1773
131	0.0657	0.0729	0.0387	0.0767	0.0764	0.0789		
132	0.0381	0.0697	0.0344	0.0288	0.0385	0.0435	0.035	0.0395
133	0.1801	0.106	0.0942	0.1019	0.1314	0.0928	0.1245	0.1771
134	0.2285	0.138	0.1311	0.1544	0.1396	0.1058	0.1331	0.1506
135	0.1956	0.1632	0.2664	0.3121	0.2425	0.1838	0.2352	0.1895
136	0.1785	0.0715	0.0681	0.0954	0.078	0.0793	0.0731	0.0575
137	0.0402	0.0427	0.0451	0.0528	0.0579	0.0433	0.0521	0.0445
138	0.0968	0.0776	0.1862	0.0827	0.0703	0.0656	0.0698	0.0703
139	0.2271	0.1824	0.1554	0.2026	0.1726	0.1537	0.1779	0.1743
140	0.2243	0.196	0.3036	0.3563	0.271	0.2359	0.3304	0.2697
141	0.2319	0.1503	0.7707	0.2096	0.1739	0.1866	0.1623	0.1019
142	0.045	0.0427	0.0538	0.0784	0.0688	0.0465	0.0577	0.0423
143	0.1588	0.1136	0.0717	0.0879	0.0734	0.0895	0.0703	0.0768
144	0.3428	0.2248	2.1224	0.3542	0.2834	0.2259	0.2775	0.2939
145	0.2455	0.2864	0.4525	0.4732	0.376	0.3252	0.3765	0.33
146	0.0539	0.0859	0.3786	0.1013	0.0666	0.0525	0.068	0.0698
147	0.0782	0.0831	0.0478	0.0462	0.0376	0.0352	0.0434	0.0443
148	0.0578	0.0293	0.0326	0.0473	0.0375	0.0289	0.0387	0.0344
149	0.0155	0.0128	0.0168	0.0095	0.0113	0.0131	0.0096	0.0184
150	0.0123	0.0087	0.1255	0.0105	0.0144	0.0074	0.0066	0.0102
151	0.1835	0.1164	0.0602	0.11	0.0928	0.0777	0.1482	0.1575
152	0.1026	0.0884	6.8359	0.0861	0.0777	0.0783	0.0785	0.099
153	0.0803	0.0971	0.0425	0.0612	0.0602	0.0538	0.0619	0.0702
154	0.0272	0.0133	0.3535	0.0177	0.0154	0.0146	0.0202	0.0177
155	0.0305	0.0231	0.2645	0.0204	0.0264	0.0225	0.0526	0.0455
156	0.1703	0.1059	0.0651	0.0925	0.062	0.0482	0.0653	0.053
157	0.0241	0.0305	0.062	0.0744	0.0497	0.0352	0.0465	0.0346
158	0.083	0.0668	1.1473	0.0923	0.0769	0.069	0.0804	0.0729

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
159	0.015	0.015	0.0584	0.0399	0.0282	0.0191	0.0517	0.0379
160	0.0244	0.0297	3.3675	0.0254	0.0356	0.0239	0.0799	0.06
161	0.2205	0.0735	0.4963	0.1435	0.107	0.0873		
162	0.0409	0.0436	0.1016	0.0385	0.0323	0.0362	0.0246	0.0316
163	0.0127	0.0127	0.086	0.0108	0.0176	0.0139	0.0142	0.0185
164	0.0172	0.0067	0.0244	0.0075	0.0058	0.0088	0.0037	0.0071
165	0.0244	0.0355	0.2955	0.0655	0.0557	0.0389	0.0508	0.0354
166	0.057	0.0211	0.0442	0.0665	0.0403	0.0252	0.0415	0.0297
167	0.1172	0.0662	0.0756	0.0572	0.0562	0.0439	0.0556	0.0591
168	0.0738	0.0704	0.4244	0.057	0.0563	0.0572	0.0547	0.061
169	0.0364	0.0157	0.015	0.0141	0.0143	0.0155	0.0163	0.0172
170	0.0787	0.0656	2.0664	0.0751	0.0709	0.0592	0.0799	0.0849
171	0.1526	0.0527	0.0557	0.09	0.0535	0.0593	0.0461	0.0303
172	0.1325	0.1148	0.1403	0.1719	0.1429	0.1135	0.1441	0.1357
173	0.0564	0.0611	0.8315	0.0385	0.0394	0.0422	0.04	0.0535
174	0.0865	0.0347	0.1228	0.0418	0.0455	0.0317	0.0471	0.0558
175	0.0267	0.0169	0.4722	0.0358	0.0253	0.0232	0.0227	0.0191
176	0.0982	0.0147	0.0498	0.0666	0.0345	0.023	0.0303	0.0184
177	0.0369	0.0454	5.3503	0.0672	0.0548	0.0386	0.0506	0.0471
178	0.0762	0.077	0.0658	0.0694	0.0731	0.0636	0.0693	0.0838
179	0.1138	0.0785	0.0689	0.1167	0.0923	0.0638	0.0938	0.0682
180	0.0455	0.0367	29.377	0.0418	0.0403	0.0355	0.0395	0.0528
181	0.2416	0.08	0.1361	0.1631	0.1162	0.0956	0.1137	0.0931
182	0.0944	0.1312	0.1316	0.1155	0.0893	0.078	0.0947	0.0802
183	0.158	0.1999	1.5615	0.2431	0.2041	0.1756	0.2453	0.2093
184	0.0298	0.0406	0.1156	0.0711	0.0508	0.0416	0.0489	0.0379
185	0.0793	0.0711	0.8593	0.0849	0.0749	0.0632	0.0821	0.0912
186	0.0131	0.0102	0.2156	0.0111	0.0127	0.0113	0.009	0.0094
187	0.0565	0.0274	0.8067	0.0203	0.0303	0.0235	0.0174	0.0255
188	0.0287	0.0243	0.1484	0.0384	0.0361	0.0221	0.0267	0.0224
189	0.0449	0.0299	0.0814	0.0568	0.0466	0.0363	0.0362	0.043
190	0.1201	0.0817	0.2921	0.0959	0.0847	0.0664	0.0882	0.1105
191	0.0188	0.0217	0.0159	0.0235	0.026	0.0177	0.0192	0.0254
192	0.0328	0.0298	0.0292	0.0249	0.0289	0.024	0.0178	0.0182
193	0.0393	0.0172	0.0102	0.0235	0.0214	0.0178	0.0158	0.01
194	0.0511	0.0488	0.0574	0.0325	0.0368	0.0388	0.0339	0.053
195	0.2312	0.1048	0.249	0.169	0.1339	0.106	0.1698	0.1577
196	0.0214	0.0188	0.0163	0.0238	0.0259	0.018	0.0199	0.0186
197	0.0325	0.0362	0.0302	0.029	0.0316	0.0276	0.0294	0.0321
198	0.1274	0.0888	0.0985	0.1043	0.0976	0.0777	0.0999	0.1141

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
199	0.0966	0.0771	0.0476	0.0613	0.0537	0.0513	0.0543	0.0601
200	0.139	0.0752	1.1184	0.1267	0.097	0.0657	0.1036	0.0894
201	0.0216	0.016	0.106	0.022	0.0171	0.0199	0.0145	0.016
202	0.1022	0.0392	0.041	0.0497	0.0493	0.0335	0.0509	0.0631
203	0.1331	0.0967	0.0975	0.0837	0.0752	0.0741	0.0769	0.0787
204	0.07	0.0949	0.1615	0.0924	0.0841	0.0709	0.0862	0.0881
205	0.1705	0.0825	1.7636	0.1664	0.1221	0.0959	0.1543	0.1363
206	0.0492	0.0235	0.0601	0.0315	0.0233	0.0286	0.0194	0.0186
207	0.1363	0.0683	0.1277	0.1562	0.1115	0.0781	0.0996	0.0901
208	0.1342	0.1304	0.1601	0.1371	0.1158	0.1184	0.1106	0.1057
209	0.2972	0.1511	0.1611	0.195	0.1718	0.1367	0.179	0.182
210	0.0737	0.0791	0.4616	0.091	0.0797	0.0719	0.0931	0.1034
211	0.0145	0.0521	3.598	0.0172	0.0218	0.0239	0.023	0.0373
212	0.0257	0.006	1.1297	0.0104	0.0068	0.0073	0.0089	0.0094
213	0.0241	0.0133	3.7502	0.0108	0.012	0.012	0.0113	0.0263
214	0.0144	0.0075	0.4475	0.0053	0.0082	0.009	0.0063	0.0056
215	0.0025	0.0026	23.891	0.0027	0.0026	0.0061	0.0198	0.0469
216	0.0408	0.0311	16.124	0.028	0.0248	0.0244	0.0252	0.0432
217	0.0171	0.0098	6.9885	0.0183	0.0123	0.0087	0.0164	0.0222
218	0.027	0.0418	4.7478	0.0212	0.0223	0.0209	0.0244	0.0352
219	0.011	0.0026	1.6913	0.0024	0.0027	0.0061	0.0049	0.012
220	0.0014	0.0016	4.3592	0.0015	0.0017	0.0043	0.0041	0.0165
221	0.0888	0.0604	11.937	0.0337	0.041	0.0454	0.0429	0.0701
222	0.0589	0.0333	83.651	0.0338	0.0325	0.0321	0.0331	0.0402
223	0.0717	0.049	2.4706	0.0533	0.0517	0.0463	0.0507	0.0493
224	0.0133	0.014	5.3849	0.0156	0.0145	0.0158	0.0434	0.0661
225	0.0019	0.002	10.014	0.0016	0.0034	0.0027		
226	0.0393	0.0164	33.806	0.028	0.0217	0.012		
227	0.0053	0.0038	2.057	0.0044	0.0055	0.007	0.0196	0.0343
228	0.0042	0.0029	3.8624	0.0031	0.0033	0.0049		
229	0.004	0.0066	1.307	0.005	0.0082	0.0077	0.0053	0.0116
230	0.0069	0.007	13.678	0.0077	0.0081	0.0059		
231	0.0186	0.0195	237.32	0.0207	0.0194	0.0198	0.0202	0.0731
232	0.0123	0.0065	0.3441	0.0062	0.008	0.0049	0.5586	0.7791
233	0.0088	0.0063	3.8289	0.0078	0.0069	0.0072	0.0073	0.0208
234	0.0251	0.0155	3.808	0.0131	0.0143	0.0127	0.015	0.0392
235	0.0034	0.0027	1.4156	0.0032	0.0035	0.0035	0.4599	0.5901
236	0.0216	0.0087	0.8692	0.0147	0.0102	0.0083	0.0115	0.0126
237	0.0211	0.0063	1.2042	0.0036	0.0036	0.0067	0.0043	0.008
238	0.0088	0.0114	5.4563	0.008	0.0094	0.0122	0.0091	0.0208

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
239	0.021	0.013	0.9279	0.0119	0.0139	0.0122	0.0134	0.0342
240	0.0138	0.0096	20.733	0.0115	0.0105	0.0087		
241	0.0261	0.0141	0.3431	0.0128	0.0117	0.0139	0.0132	0.0185
242	0.0228	0.0141	12.573	0.0191	0.0157	0.0172	0.0167	0.0147
243	0.0279	0.0205	0.1492	0.0182	0.0175	0.0227	0.019	0.027
244	0.0241	0.0232	0.7206	0.0299	0.0249	0.0229	0.0266	0.0222
245	0.0093	0.0086	8.2542	0.0098	0.0082	0.0127		
246	0.0457	0.0129	4.4251	0.0128	0.0111	0.0137	0.0792	0.086
247	0.021	0.0389	72.995	0.0196	0.0249	0.0279	0.025	0.051
248	0.0422	0.0518	10.336	0.0595	0.0546	0.0436	0.0542	0.0704
249	0.0217	0.0296	0.9677	0.0296	0.0294	0.0285		
250	0.0208	0.0186	22.934	0.0204	0.0176	0.0174	0.0193	0.0423
251	0.0082	0.0103	53.814	0.0114	0.0119	0.0091	0.0097	0.0363
252	0.008	0.0026	0.3887	0.0022	0.004	0.0053	0.0084	0.0071
253	0.0049	0.0039	0.6941	0.0048	0.0044	0.0049	0.0033	0.0162
254	0.0191	0.0238	16.016	0.027	0.0242	0.0231	0.0256	0.0509
255	0.0269	0.0105	4.7077	0.0124	0.0105	0.0111	0.0158	0.0301
256	0.0065	0.0071	31.636	0.0074	0.0099	0.0046		
257	0.0146	0.0109	2.0432	0.0109	0.0099	0.0134	0.011	0.0202
258	0.0209	0.012	6.0896	0.0141	0.0139	0.011	0.014	0.0227
259	0.0312	0.0163	1.339	0.017	0.0175	0.0174	0.0175	0.0339
260	0.0449	0.0298	15.56	0.0352	0.031	0.0223	0.0357	0.0694
261	0.0153	0.0073	0.2154	0.0085	0.0088	0.0066	0.0073	0.0112
262	0.0125	0.0097	3.2053	0.0085	0.0105	0.0122	0.0095	0.0126
263	0.0288	0.0206	0.4653	0.0276	0.0246	0.0187	0.0234	0.0177
264	0.0358	0.0398	4.3134	0.0352	0.0372	0.0423	0.0664	0.1157
265	0.0522	0.0319	15.313	0.0431	0.0356	0.0281	0.0354	0.0616
266	0.019	0.0088	2.048	0.0093	0.0081	0.0121	0.012	0.0232
267	0.0179	0.0087	3.2741	0.0115	0.0083	0.007	0.0087	0.0141
268	0.0294	0.0212	2.1793	0.0224	0.0202	0.0248	0.0234	0.0278
269	0.0303	0.0241	2.9216	0.0239	0.0226	0.017	0.0352	0.052
270	0.0654	0.0386	5.577	0.0471	0.0415	0.0376	1.7084	2.2163
271	0.0057	0.005	12.718	0.0069	0.0062	0.0076		
272	0.0398	0.0136	0.3651	0.0155	0.0135	0.0111	0.071	0.0725
273	0.041	0.0356	4.7282	0.034	0.033	0.0371	0.0355	0.0404
274	0.056	0.0641	160.26	0.0668	0.0636	0.0547		
275	0.0455	0.0499	654.39	0.0462	0.0468	0.0444	0.0489	0.1541

Appendix H: The Sample Analysis Set MSRE Results

Sample Size 15

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
1	0.0703	0.0185	0.0187	0.0291	0.026	0.0537	0.0245	0.0225
2	0.041	0.0238	0.016	0.0225	0.0192	0.0305	0.0184	0.0188
3	0.0226	0.0158	0.0094	0.0139	0.01	0.0108	0.0099	0.0103
4	0.0337	0.0174	0.0121	0.0165	0.011	0.0169	0.0104	0.0104
5	0.0221	0.0081	0.0107	0.0114	0.0109	0.02	0.0098	0.0091
6	0.0525	0.0318	0.0183	0.0213	0.0221	0.0393	0.0216	0.0207
7	0.0197	0.0152	0.0086	0.0095	0.0077	0.0083	0.0076	0.0076
8	0.0193	0.0166	0.0067	0.0072	0.0068	0.0077	0.007	0.0072
9	0.0209	0.0151	0.0114	0.0117	0.0113	0.0103	0.0114	0.0111
10	0.0316	0.0134	0.015	0.0191	0.0154	0.0244	0.0145	0.014
11	0.0473	0.0302	0.0203	0.0274	0.0218	0.0326	0.0216	0.0215
12	0.0351	0.0275	0.0149	0.0151	0.0147	0.0148	0.0144	0.0146
13	0.0135	0.016	0.0092	0.01	0.0094	0.0087	0.0094	0.0095
14	0.0308	0.0313	0.0196	0.0203	0.0186	0.0193	0.0186	0.0186
15	0.0475	0.0143	0.0106	0.0127	0.0108	0.0232	0.0103	0.0096
16	0.0257	0.0168	0.0072	0.0126	0.0103	0.0242	0.0095	0.0083
17	0.0234	0.0196	0.0132	0.0128	0.013	0.0149	0.0131	0.0129
18	0.0215	0.0189	0.0096	0.0107	0.0091	0.0095	0.0091	0.0091
19	0.0211	0.0139	0.0116	0.0146	0.0105	0.0108	0.0105	0.0104
20	0.0154	0.0103	0.0097	0.0125	0.0108	0.0124	0.01	0.01
21	0.0634	0.0301	0.0157	0.0267	0.0211	0.0471	0.0198	0.0177
22	0.038	0.0248	0.011	0.0144	0.0131	0.0249	0.012	0.0112
23	0.0274	0.014	0.0136	0.0134	0.0127	0.0158	0.0125	0.0121
24	0.0292	0.0183	0.0132	0.0169	0.0132	0.0167	0.013	0.013
25	0.0381	0.0215	0.0169	0.0212	0.0181	0.0315	0.0171	0.0163
26	0.0277	0.0223	0.0187	0.0262	0.0158	0.0273	0.0136	0.0132
27	0.019	0.012	0.0083	0.0074	0.0065	0.0131	0.0063	0.0064
28	0.0093	0.0052	0.0062	0.0088	0.0049	0.0032	0.0049	0.0055
29	0.0116	0.0086	0.0068	0.0069	0.0028	0.0046	0.0025	0.0026
30	0.0109	0.0096	0.0139	0.0203	0.0135	0.0204	0.0097	0.009
31	0.0282	0.0166	0.0088	0.0115	0.0081	0.0167	0.0068	0.0064
32	0.0267	0.0162	0.0125	0.0153	0.0091	0.0121	0.0088	0.0085
33	0.0272	0.0201	0.014	0.0172	0.0142	0.0141	0.0147	0.0153
34	0.0054	0.0118	0.0043	0.007	0.006	0.005	0.0066	0.0065
35	0.029	0.0095	0.0184	0.0209	0.0149	0.0219	0.0126	0.0119
36	0.0196	0.0132	0.0156	0.0205	0.0113	0.0216	0.0096	0.0089
37	0.0115	0.0066	0.0069	0.011	0.0058	0.0048	0.0059	0.0057

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
38	0.0169	0.0349	0.0102	0.0089	0.0101	0.0134	0.01	0.01
39	0.0196	0.0138	0.0136	0.0217	0.0155	0.0126	0.0165	0.0153
40	0.0143	0.0101	0.0115	0.0114	0.008	0.0093	0.007	0.0072
41	0.0386	0.0217	0.0225	0.0298	0.018	0.0219	0.0164	0.0159
42	0.0352	0.0262	0.0188	0.0227	0.0169	0.0184	0.0165	0.0164
43	0.018	0.0212	0.015	0.0216	0.0136	0.0143	0.0136	0.0133
44	0.0314	0.0174	0.0211	0.0302	0.0176	0.0188	0.0177	0.0166
45	0.0182	0.0105	0.0249	0.0331	0.0182	0.0259	0.0157	0.0144
46	0.053	0.0248	0.0365	0.0625	0.033	0.0461	0.0282	0.0253
47	0.0326	0.028	0.0317	0.0378	0.0222	0.0309	0.0191	0.0173
48	0.0154	0.0075	0.0166	0.0208	0.0123	0.0142	0.0106	0.0095
49	0.025	0.0133	0.0188	0.0224	0.013	0.0182	0.011	0.01
50	0.0159	0.0116	0.0175	0.025	0.0169	0.0194	0.0144	0.014
51	0.0203	0.0177	0.0116	0.0152	0.0117	0.0117	0.0095	0.0097
52	0.0061	0.005	0.0055	0.0049	0.0039	0.0051	0.0034	0.0034
53	0.0069	0.0056	0.0061	0.0051	0.0041	0.0052	0.004	0.004
54	0.0062	0.0038	0.006	0.005	0.004	0.0044	0.0035	0.0035
55	0.0105	0.004	0.0072	0.0094	0.0058	0.006	0.0041	0.0041
56	0.0108	0.0131	0.0099	0.0122	0.008	0.0071	0.0071	0.0071
57	0.0088	0.0087	0.0042	0.0053	0.0041	0.0048	0.0043	0.0043
58	0.0114	0.0093	0.0067	0.0065	0.0065	0.0086	0.0064	0.0067
59	0.0089	0.0098	0.0084	0.0069	0.0068	0.0091	0.0066	0.0067
60	0.0084	0.0056	0.0136	0.0184	0.0107	0.0089	0.0074	0.0068
61	0.0162	0.0173	0.0125	0.0167	0.0112	0.0079	0.0093	0.0093
62	0.012	0.0136	0.0071	0.0091	0.006	0.0076	0.0062	0.0061
63	0.0106	0.0108	0.0071	0.0101	0.0063	0.0067	0.0065	0.0061
64	0.0114	0.0085	0.0054	0.0072	0.0054	0.0055	0.0058	0.0055
65	0.0079	0.0112	0.0112	0.0148	0.0115	0.012	0.0087	0.0083
66	0.0226	0.012	0.0153	0.0198	0.0096	0.0104	0.0095	0.0093
67	0.021	0.0181	0.0163	0.0218	0.0113	0.0113	0.0111	0.0109
68	0.0078	0.0208	0.0088	0.0096	0.009	0.0109	0.0087	0.0092
69	0.0144	0.0077	0.0185	0.0253	0.0128	0.0101	0.013	0.0115
70	0.0031	0.0041	0.007	0.0084	0.0052	0.0041	0.005	0.0051
71	0.0137	0.0151	0.012	0.0187	0.01	0.0096	0.0093	0.0092
72	0.0112	0.0126	0.0142	0.0177	0.0095	0.0076	0.0092	0.0083
73	0.0174	0.0205	0.0259	0.0292	0.0193	0.0163	0.0183	0.017
74	0.0208	0.0223	0.0259	0.0304	0.0197	0.0165	0.018	0.0167
75	0.0082	0.0076	0.0132	0.0198	0.0115	0.0095	0.008	0.0072
76	0.0026	0.0066	0.1382	0.0035	0.0045	0.0053	0.0032	0.0033
77	0.0023	0.0013	0.0581	0.0008	0.0011	0.0027	0.001	0.0009
78	0.0013	0.0013	0.0609	0.0013	0.0012	0.0048	0.0011	0.002

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
79	0.0009	0.0008	0.0365	0.0009	0.0009	0.003	0.0008	0.001
80	0.0031	0.0028	0.155	0.004	0.0036	0.0039	0.003	0.0029
81	0.0026	0.002	0.1665	0.0013	0.0014	0.0022	0.0011	0.0012
82	0.0022	0.0022	0.0392	0.0012	0.0012	0.0037	0.0012	0.0015
83	0.0012	0.0012	0.012	0.0013	0.0011	0.0029	0.0012	0.0015
84	0.0026	0.0011	0.0142	0.0012	0.0011	0.0025	0.0011	0.0011
85	0.0016	0.001	0.0784	0.001	0.0011	0.0022	0.0007	0.0007
86	0.0034	0.003	0.0494	0.0033	0.0028	0.0034	0.0026	0.0026
87	0.0016	0.0025	0.0396	0.0013	0.0012	0.0045	0.0011	0.0019
88	0.0027	0.0023	0.0168	0.0021	0.002	0.0043	0.002	0.0026
89	0.0036	0.0024	0.0232	0.003	0.0024	0.0048	0.0026	0.0025
90	0.0033	0.0019	0.0692	0.0024	0.0022	0.0027	0.0015	0.0015
91	0.0042	0.0033	0.1033	0.002	0.0019	0.0046	0.002	0.0021
92	0.0042	0.0036	0.0169	0.0025	0.0022	0.0051	0.0022	0.0028
93	0.0018	0.004	0.0081	0.0015	0.0023	0.0048	0.002	0.0034
94	0.0047	0.0035	0.0265	0.0022	0.0024	0.0046	0.0023	0.0026
95	0.0027	0.0035	0.1121	0.004	0.0035	0.0044	0.0031	0.0031
96	0.0046	0.0026	0.1936	0.0029	0.002	0.003	0.0019	0.0019
97	0.0072	0.0053	0.027	0.0037	0.0031	0.0049	0.0031	0.0034
98	0.0129	0.0062	0.0606	0.0117	0.0082	0.0074	0.0087	0.0073
99	0.0043	0.0049	0.0322	0.006	0.0045	0.0046	0.0045	0.0041
100	0.0019	0.0029	0.1441	0.0042	0.0034	0.0029	0.0028	0.0026

Sample Size = 9

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
1	0.067	0.0312	0.0409	0.0576	0.0536	0.0752	0.052	0.0421
2	0.0493	0.0237	0.0081	0.0129	0.0139	0.0323	0.0129	0.0128
3	0.0322	0.0162	0.0086	0.0107	0.0078	0.009	0.0077	0.0087
4	0.0453	0.0408	0.0181	0.0224	0.0186	0.0317	0.0185	0.019
5	0.0713	0.0346	0.02	0.0241	0.0221	0.0378	0.0208	0.0175
6	0.0434	0.0247	0.0163	0.0206	0.018	0.0305	0.0173	0.0159
7	0.0265	0.0395	0.0213	0.0219	0.0211	0.025	0.0211	0.0207
8	0.0131	0.0114	0.0047	0.0047	0.0047	0.0052	0.0047	0.0048
9	0.0313	0.0151	0.0122	0.0134	0.0128	0.0117	0.013	0.013
10	0.0238	0.0162	0.0153	0.0174	0.0149	0.0191	0.0144	0.0158
11	0.034	0.0351	0.0224	0.0262	0.0209	0.0296	0.0208	0.0249
12	0.0326	0.0241	0.019	0.0196	0.0175	0.0182	0.0172	0.0181
13	0.0228	0.0211	0.0126	0.0138	0.0127	0.014	0.0127	0.012
14	0.0305	0.0412	0.0227	0.0232	0.023	0.0276	0.0233	0.0229
15	0.0333	0.0377	0.0227	0.0344	0.0336	0.0451	0.0304	0.0236
16	0.0926	0.0468	0.0213	0.0289	0.0259	0.0447	0.0244	0.0199
17	0.0396	0.0458	0.0257	0.0262	0.0258	0.0242	0.0258	0.0295
18	0.0368	0.0302	0.0153	0.0166	0.0156	0.016	0.0157	0.0161
19	0.0598	0.0468	0.0238	0.0258	0.0247	0.0293	0.0252	0.024
20	0.0665	0.0232	0.025	0.0311	0.0302	0.0398	0.0289	0.0256
21	0.1027	0.0522	0.036	0.0626	0.0551	0.082	0.0529	0.0371
22	0.0579	0.0475	0.0248	0.0233	0.026	0.0473	0.025	0.019
23	0.0455	0.0243	0.0136	0.016	0.0137	0.0183	0.0135	0.0144
24	0.0623	0.0302	0.0443	0.0514	0.0423	0.0472	0.0405	0.0305
25	0.0535	0.0389	0.028	0.0326	0.0277	0.0394	0.0275	0.0288
26	0.0762	0.0454	0.0347	0.0493	0.0387	0.0552	0.0339	0.0312
27	0.0244	0.0109	0.0132	0.015	0.0182	0.0164	0.0158	0.0163
28	0.0266	0.0258	0.0245	0.0247	0.0228	0.0209	0.0225	0.0252
29	0.0247	0.0186	0.0099	0.0106	0.0082	0.011	0.0077	0.0084
30	0.0208	0.012	0.01	0.0137	0.0105	0.0167	0.0087	0.0086
31	0.0397	0.0141	0.0245	0.035	0.0216	0.0265	0.0178	0.0128
32	0.0211	0.02	0.014	0.0135	0.0147	0.0132	0.015	0.0157
33	0.0323	0.0295	0.0203	0.0202	0.02	0.0201	0.0201	0.0203
34	0.0298	0.0258	0.0173	0.0189	0.0128	0.017	0.0123	0.0118
35	0.0522	0.0152	0.0215	0.0253	0.0181	0.0243	0.0163	0.0142
36	0.0219	0.0119	0.0165	0.0251	0.0147	0.0206	0.012	0.009
37	0.0348	0.0287	0.0165	0.02	0.0149	0.0171	0.0149	0.0146
38	0.0118	0.0228	0.0096	0.0092	0.0091	0.0122	0.009	0.0088
39	0.0279	0.0148	0.0058	0.0106	0.0068	0.0025	0.0071	0.0051

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
40	0.034	0.0246	0.0331	0.0379	0.023	0.0299	0.0217	0.0232
41	0.0216	0.0208	0.0129	0.016	0.0118	0.0089	0.0125	0.0158
42	0.0254	0.0221	0.022	0.0296	0.0157	0.0142	0.0146	0.0144
43	0.0369	0.0309	0.0257	0.037	0.0227	0.0214	0.023	0.02
44	0.0534	0.0386	0.0284	0.0352	0.0237	0.0297	0.0233	0.0222
45	0.0141	0.015	0.0213	0.0224	0.0191	0.0167	0.0183	0.0212
46	0.0725	0.0512	0.0636	0.0999	0.0673	0.0604	0.0641	0.0559
47	0.0538	0.0373	0.0438	0.0589	0.0365	0.0378	0.0341	0.0316
48	0.0559	0.0473	0.0379	0.0395	0.0332	0.0368	0.0321	0.0325
49	0.051	0.0341	0.0198	0.0253	0.0191	0.0256	0.018	0.0158
50	0.0542	0.0278	0.0175	0.025	0.0185	0.0238	0.0157	0.0163
51	0.0194	0.0076	0.019	0.0279	0.0165	0.0113	0.0119	0.01
52	0.0186	0.0122	0.0099	0.0094	0.0107	0.0098	0.0096	0.0099
53	0.0082	0.0047	0.005	0.0053	0.0046	0.0048	0.0044	0.0057
54	0.0074	0.0051	0.0085	0.0076	0.007	0.0065	0.0061	0.0066
55	0.0423	0.0208	0.0405	0.0494	0.0354	0.0302	0.0345	0.0321
56	0.0247	0.0144	0.0169	0.0194	0.0134	0.0123	0.012	0.0121
57	0.0187	0.0157	0.0131	0.0132	0.0132	0.0129	0.013	0.013
58	0.0058	0.0062	0.0032	0.003	0.0035	0.0044	0.0035	0.0038
59	0.0163	0.0134	0.0179	0.0217	0.017	0.0144	0.0171	0.0161
60	0.0137	0.0109	0.0265	0.0327	0.0193	0.017	0.0161	0.0137
61	0.0222	0.0237	0.0127	0.013	0.0129	0.0138	0.0135	0.0161
62	0.0097	0.006	0.0048	0.0071	0.0047	0.0039	0.0048	0.0041
63	0.0222	0.0108	0.0157	0.022	0.0147	0.01	0.0156	0.015
64	0.0402	0.0253	0.0252	0.0315	0.0235	0.021	0.0244	0.0226
65	0.0117	0.01	0.0147	0.0191	0.0115	0.0114	0.0085	0.007
66	0.0463	0.0238	0.0324	0.0402	0.0249	0.0221	0.0224	0.019
67	0.0426	0.021	0.0264	0.0323	0.0206	0.0178	0.0212	0.0193
68	0.0232	0.0198	0.0175	0.0213	0.0153	0.0154	0.0155	0.0152
69	0.0457	0.0328	0.031	0.0369	0.0247	0.0247	0.0253	0.0225
70	0.0402	0.0209	0.0552	0.0662	0.0436	0.0348	0.0398	0.0325
71	0.0344	0.0142	0.0424	0.06	0.0297	0.0223	0.0264	0.0187
72	0.0349	0.027	0.0285	0.0319	0.022	0.0232	0.0201	0.0182
73	0.0404	0.0341	0.0474	0.0553	0.0355	0.0368	0.0351	0.0288
74	0.0383	0.0274	0.0462	0.0527	0.0335	0.03	0.0322	0.0261
75	0.0808	0.023	0.0603	0.075	0.0463	0.0433	0.0457	0.0389
76	0.0045	0.0054	0.178	0.0042	0.0042	0.0052	0.0038	0.0044
77	0.0036	0.0021	0.1197	0.0023	0.0021	0.0046	0.0019	0.0024
78	0.0047	0.0023	0.0342	0.0024	0.0026	0.0045	0.0026	0.0039
79	0.0022	0.0019	0.065	0.0025	0.0025	0.0038	0.0022	0.0031
80	0.0096	0.0057	0.2448	0.0083	0.0071	0.0057	0.0059	0.0064

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
81	0.006	0.0039	0.0816	0.0038	0.0032	0.0035	0.0029	0.0033
82	0.0045	0.0025	0.068	0.0029	0.0025	0.0044	0.0024	0.0026
83	0.0017	0.0017	0.0218	0.0014	0.0013	0.0037	0.0013	0.0013
84	0.0118	0.0097	0.0994	0.0128	0.0111	0.0105	0.011	0.0117
85	0.004	0.0036	0.1876	0.0039	0.0036	0.0049	0.0039	0.0047
86	0.0106	0.0053	0.0776	0.0063	0.0053	0.0049	0.0054	0.0053
87	0.0061	0.008	0.0479	0.0072	0.0069	0.0098	0.007	0.0074
88	0.0124	0.0102	0.0166	0.0117	0.0106	0.0093	0.011	0.0109
89	0.0068	0.0051	0.0316	0.0038	0.0037	0.0063	0.0038	0.0045
90	0.0061	0.0057	0.6476	0.0073	0.0067	0.006	0.006	0.0065
91	0.0121	0.0069	0.132	0.0109	0.0084	0.0062	0.0083	0.008
92	0.0127	0.0095	0.0546	0.0103	0.0091	0.0112	0.0091	0.0095
93	0.0135	0.0068	0.0478	0.0121	0.0088	0.0066	0.0094	0.0077
94	0.0215	0.0129	0.0844	0.0165	0.014	0.0148	0.0142	0.0137
95	0.0174	0.0085	0.0939	0.0116	0.0099	0.0096	0.0101	0.01
96	0.0055	0.0033	0.1442	0.0045	0.0032	0.0043	0.0033	0.0036
97	0.0094	0.0114	0.0689	0.0118	0.0099	0.0096	0.0101	0.0097
98	0.0088	0.0098	0.0334	0.0155	0.0117	0.0099	0.0121	0.0103
99	0.0121	0.0115	0.075	0.0141	0.012	0.0105	0.0119	0.0106
100	0.0149	0.0104	0.3309	0.0154	0.0127	0.0101	0.0121	0.0111

Sample Size = 6

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
1	0.0807	0.0622	0.0373	0.0564	0.051	0.0727	0.0476	0.0364
2	0.1057	0.0588	0.0297	0.0335	0.0357	0.0542	0.0343	0.0328
3	0.0306	0.027	0.0173	0.0204	0.0227	0.0285	0.0214	0.026
4	0.0349	0.0308	0.014	0.012	0.0156	0.0215	0.0154	0.0159
5	0.0748	0.0574	0.0411	0.0646	0.059	0.0764	0.0558	0.0432
6	0.0803	0.0731	0.0527	0.0702	0.0609	0.082	0.058	0.0487
7	0.0224	0.0347	0.0193	0.0205	0.02	0.0227	0.0205	0.0192
8	0.0255	0.0236	0.0209	0.0232	0.0218	0.0207	0.0222	0.0235
9	0.013	0.0236	0.0136	0.0122	0.0128	0.0148	0.0125	0.0136
10	0.029	0.0146	0.0168	0.0172	0.0169	0.0138	0.0168	0.0155
11	0.0267	0.0227	0.0104	0.0141	0.015	0.0236	0.0142	0.0117
12	0.0352	0.0363	0.0296	0.0302	0.03	0.0292	0.0301	0.0302
13	0.0121	0.0064	0.0068	0.0085	0.0068	0.005	0.0069	0.0059
14	0.0524	0.0745	0.0396	0.0401	0.0397	0.0464	0.0398	0.0395
15	0.0676	0.0527	0.0416	0.0474	0.0376	0.0436	0.0369	0.0459
16	0.0907	0.0701	0.0577	0.0713	0.0623	0.0712	0.0608	0.0537
17	0.039	0.0414	0.0319	0.0327	0.0325	0.0322	0.0327	0.0352
18	0.0438	0.0449	0.0347	0.0344	0.0353	0.0377	0.0351	0.0381
19	0.0398	0.0555	0.0425	0.0409	0.0412	0.0439	0.0412	0.0401
20	0.11	0.0867	0.0464	0.0598	0.055	0.0889	0.0535	0.0395
21	0.1346	0.0704	0.0579	0.1054	0.0905	0.1205	0.0868	0.0528
22	0.0518	0.0366	0.0288	0.0367	0.0357	0.049	0.0345	0.0228
23	0.059	0.0431	0.0371	0.0416	0.0378	0.0438	0.037	0.0342
24	0.0816	0.0647	0.0415	0.0473	0.0416	0.0562	0.0403	0.0368
25	0.0728	0.0532	0.0328	0.0446	0.0361	0.0525	0.0355	0.0326
26	0.0681	0.0542	0.0429	0.0479	0.0385	0.0541	0.0361	0.0368
27	0.0222	0.0228	0.0166	0.0177	0.0147	0.0167	0.0139	0.0134
28	0.0123	0.0243	0.0183	0.0142	0.017	0.0182	0.0171	0.0207
29	0.0229	0.0197	0.0101	0.0092	0.0085	0.0111	0.0083	0.0086
30	0.0763	0.0384	0.0447	0.0615	0.0456	0.0491	0.0398	0.0342
31	0.0208	0.0215	0.0254	0.0302	0.0198	0.0206	0.0187	0.02
32	0.0243	0.0215	0.0103	0.0126	0.0136	0.0129	0.0141	0.0134
33	0.0038	0.006	0.0065	0.0054	0.0056	0.0054	0.0055	0.0056
34	0.025	0.0199	0.0169	0.0173	0.0136	0.0166	0.0134	0.013
35	0.1182	0.0887	0.0732	0.087	0.0662	0.0947	0.0614	0.05
36	0.007	0.0062	0.0099	0.0098	0.0085	0.0067	0.0093	0.0124
37	0.0252	0.0235	0.0164	0.0178	0.0163	0.0175	0.0165	0.0165
38	0.0072	0.0056	0.0051	0.0067	0.005	0.0041	0.0051	0.005
39	0.0358	0.0478	0.0137	0.0214	0.0126	0.0085	0.0131	0.0088

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
40	0.0331	0.0325	0.0266	0.0298	0.0286	0.0297	0.0255	0.0261
41	0.0681	0.0703	0.0626	0.0857	0.0644	0.0675	0.0607	0.0492
42	0.0776	0.0767	0.0465	0.0504	0.0477	0.0517	0.0483	0.0508
43	0.0327	0.0388	0.0226	0.0249	0.0229	0.0215	0.0229	0.0251
44	0.0374	0.0352	0.0178	0.0198	0.0177	0.0238	0.0174	0.0179
45	0.0533	0.0393	0.0227	0.0311	0.0235	0.0319	0.0217	0.0208
46	0.0679	0.0349	0.0331	0.0534	0.0353	0.039	0.0311	0.0286
47	0.0806	0.068	0.0458	0.0581	0.0437	0.0465	0.0416	0.038
48	0.0846	0.0725	0.0562	0.0644	0.0546	0.0597	0.0517	0.044
49	0.0684	0.0571	0.042	0.0481	0.0364	0.0451	0.0336	0.0309
50	0.0536	0.0441	0.04	0.0626	0.0377	0.0514	0.033	0.0211
51	0.0316	0.0246	0.0339	0.0454	0.0341	0.026	0.029	0.0258
52	0.0207	0.0139	0.0173	0.0181	0.0163	0.0138	0.0149	0.0157
53	0.0116	0.0111	0.0128	0.0114	0.011	0.0124	0.0109	0.0115
54	0.0105	0.0069	0.0067	0.0059	0.0058	0.0074	0.0056	0.0053
55	0.031	0.0194	0.0293	0.035	0.026	0.0242	0.0219	0.02
56	0.0283	0.0158	0.0303	0.037	0.0258	0.0194	0.0237	0.0193
57	0.0064	0.0044	0.008	0.0088	0.0061	0.0061	0.0061	0.0058
58	0.0106	0.0124	0.0149	0.0162	0.0133	0.0137	0.0133	0.0135
59	0.0137	0.0125	0.0133	0.014	0.0122	0.012	0.0122	0.0112
60	0.0983	0.0508	0.081	0.0935	0.0704	0.0628	0.0688	0.0613
61	0.0204	0.0191	0.0125	0.0148	0.011	0.0105	0.0101	0.0109
62	0.0204	0.0199	0.0162	0.0182	0.0154	0.0154	0.0156	0.0158
63	0.0103	0.012	0.0107	0.0131	0.0098	0.0086	0.0105	0.0108
64	0.0236	0.0351	0.0241	0.0267	0.0202	0.0231	0.0202	0.02
65	0.0601	0.0403	0.0536	0.0589	0.0431	0.04	0.041	0.0356
66	0.0371	0.0287	0.0338	0.0403	0.0269	0.023	0.0241	0.0207
67	0.0153	0.024	0.02	0.0247	0.0173	0.0142	0.0174	0.0161
68	0.0535	0.026	0.0437	0.0521	0.0377	0.0307	0.0384	0.0326
69	0.0408	0.0277	0.047	0.0554	0.0363	0.0306	0.0368	0.0302
70	0.0664	0.0387	0.082	0.0909	0.0627	0.0566	0.0637	0.0588
71	0.0422	0.0401	0.0579	0.081	0.047	0.0372	0.0408	0.0282
72	0.0394	0.0306	0.0486	0.0566	0.0396	0.0326	0.0391	0.0335
73	0.0658	0.036	0.0692	0.077	0.051	0.0421	0.0498	0.0381
74	0.0868	0.054	0.1117	0.1217	0.086	0.0707	0.0834	0.0637
75	0.0454	0.0348	0.0428	0.0489	0.0372	0.0318	0.0361	0.0371
76	0.0039	0.0046	0.5022	0.0068	0.0056	0.0051	0.0047	0.0056
77	0.0069	0.0076	0.0486	0.007	0.0074	0.0097	0.0074	0.0085
78	0.0028	0.0029	0.2378	0.0027	0.0027	0.0048	0.0028	0.0056
79	0.0088	0.0081	0.0874	0.0091	0.0087	0.0088	0.0085	0.0121
80	0.0295	0.0231	0.2918	0.0355	0.0298	0.0237	0.2162	0.2175

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
81	0.0037	0.0051	0.2586	0.0053	0.0045	0.0048	0.0058	0.0078
82	0.0047	0.0024	0.0777	0.0034	0.003	0.0031	0.0039	0.0051
83	0.0098	0.0064	0.0433	0.0067	0.0067	0.0064	0.0085	0.0126
84	0.0053	0.0042	0.4705	0.0046	0.0045	0.005	0.0057	0.0091
85	0.0248	0.017	0.2674	0.0269	0.0216	0.0162	0.0279	0.0302
86	0.0128	0.0137	0.1927	0.0164	0.0144	0.0131	0.0176	0.0189
87	0.0133	0.0081	0.3237	0.0121	0.0097	0.0087	0.0128	0.0138
88	0.0041	0.0053	0.285	0.0045	0.0046	0.0072	0.0057	0.0062
89	0.055	0.046	1.8768	0.0559	0.0511	0.0418	0.0513	0.0565
90	0.013	0.0085	0.1541	0.0114	0.01	0.0086	0.0096	0.0094
91	0.0115	0.0069	0.2842	0.0099	0.008	0.0071	0.0082	0.0094
92	0.0311	0.0216	1.849	0.0251	0.0229	0.0229	0.0232	0.0266
93	0.0488	0.0299	0.554	0.0407	0.0352	0.0295	0.0364	0.0341
94	0.0461	0.0216	0.4577	0.0336	0.0276	0.0218	0.0289	0.0283
95	0.0135	0.014	1.0797	0.0179	0.0158	0.0128	0.0156	0.0169
96	0.0136	0.0086	0.2007	0.0121	0.0092	0.0095	0.0098	0.0096
97	0.0305	0.0181	0.3483	0.0299	0.0226	0.0192	0.0239	0.0203
98	0.0292	0.0159	0.1131	0.0279	0.0212	0.0151	0.0226	0.018
99	0.0522	0.0288	0.1919	0.0431	0.036	0.0293	0.0376	0.033
100	0.0207	0.0117	2.7041	0.0205	0.0162	0.0116	0.0407	0.0384

Sample Size = 3

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
1	0.2196	0.1244	0.1225	0.187	0.1716	0.1535		
2	0.0989	0.0648	0.0477	0.0593	0.0614	0.0642	0.1059	0.1072
3	0.0341	0.0476	0.0338	0.0359	0.0351	0.0397		
4	0.0289	0.0251	0.0171	0.0204	0.02	0.0204		
5	0.1026	0.0729	0.0531	0.0799	0.073	0.0762	0.1368	0.1003
6	0.0546	0.0298	0.015	0.0195	0.0184	0.0219	0.0329	0.0285
7	0.0042	0.0063	0.004	0.0038	0.0038	0.0054	0.0069	0.0073
8	0.0063	0.0021	0.0028	0.0032	0.003	0.0019	0.0057	0.0053
9	0.0149	0.0101	0.0081	0.0081	0.0081	0.0089	0.015	0.016
10	0.034	0.0228	0.0183	0.0204	0.0203	0.0232	0.0359	0.028
11	0.0411	0.0362	0.0274	0.0291	0.0321	0.0321	0.057	0.0455
12	0.0196	0.0121	0.0102	0.0098	0.0107	0.0118	0.0199	0.0208
13	0.0001	0.0009	0.0002	0.0001	0.0002	0.0006	0.0003	0.0005
14	0.0548	0.0464	0.0479	0.0556	0.0541	0.0371	0.0567	0.0382
15	0.1191	0.0754	0.0546	0.0688	0.056	0.0555	0.0558	0.056
16	0.1359	0.1379	0.0695	0.097	0.0865	0.1077	0.0831	0.0675
17	0.0258	0.0247	0.019	0.0206	0.0181	0.02	0.0183	0.0162
18	0.021	0.0121	0.0089	0.0094	0.0092	0.0111	0.0091	0.0106
19	0.0091	0.0149	0.0142	0.0156	0.014	0.0123	0.0142	0.0125
20	0.0685	0.0477	0.0473	0.0729	0.0628	0.0621	0.0518	0.0333
21	0.3268	0.1324	0.1107	0.1753	0.1664	0.1592	0.1618	0.1042
22	0.2474	0.1836	0.1046	0.1524	0.1433	0.1614		
23	0.0181	0.0099	0.0184	0.0166	0.0188	0.0141	0.0186	0.0158
24	0.1599	0.152	0.1061	0.1163	0.1055	0.1203	0.1035	0.1062
25	0.1206	0.0937	0.1441	0.2047	0.1777	0.1407	0.1696	0.121
26	0.0983	0.08	0.0811	0.0901	0.0937	0.0832		
27	0.0632	0.0563	0.0458	0.0515	0.0566	0.0514		
28	0.0295	0.0373	0.032	0.0319	0.0345	0.0332		
29	0.0242	0.0345	0.0198	0.0175	0.018	0.0231	0.0714	0.0753
30	0.1337	0.0881	0.0805	0.1039	0.0826	0.0868	0.0823	0.0706
31	0.0476	0.0364	0.0359	0.0399	0.0381	0.0362	0.1357	0.1159
32	0.0168	0.014	0.0083	0.0084	0.0092	0.0097	0.0354	0.034
33	0.0018	0.004	0.003	0.0027	0.0026	0.0036	0.0098	0.0089
34	0.007	0.0053	0.005	0.0043	0.0044	0.005	0.0176	0.0209
35	0.0084	0.0088	0.0084	0.0099	0.009	0.0086	0.0289	0.024
36	0.0374	0.0219	0.0123	0.015	0.0137	0.0106	0.0521	0.0472
37	0.0147	0.0075	0.0036	0.0036	0.0042	0.0055	0.0167	0.0179
38	0.0079	0.0212	0.0104	0.0094	0.0104	0.0147	0.0394	0.0401
39	0.0015	0.0632	0.0322	0.0211	0.0329	0.0581	0.0318	0.0395

Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
40	0.0356	0.0921	0.0945	0.1146	0.0848	0.0556	0.0744	0.0585
41	0.1716	0.1017	0.087	0.094	0.0832	0.0946	0.0855	0.0932
42	0.0613	0.0608	0.0753	0.0891	0.0735	0.0574	0.0738	0.0634
43	0.0728	0.0785	0.0874	0.0957	0.0845	0.0759	0.0844	0.0802
44	0.0207	0.0432	0.0267	0.0299	0.0257	0.0198	0.0256	0.0253
45	0.1279	0.1216	0.111	0.1083	0.1155	0.1164	0.113	0.0997
46	0.3177	0.1651	0.1204	0.1805	0.1204	0.1296	0.1151	0.089
47	0.0991	0.0827	0.1195	0.1566	0.11	0.0912		
48	0.1783	0.0841	0.0748	0.1039	0.0717	0.0624	0.0737	0.059
49	0.1877	0.1035	0.0822	0.0989	0.0793	0.0749	0.076	0.0693
50	0.1533	0.0843	0.078	0.1107	0.0815	0.0707	0.0873	0.0726
51	0.1183	0.0533	0.0741	0.0882	0.0671	0.0578	0.0633	0.0563
52	0.0528	0.0368	0.0324	0.0379	0.0444	0.0314		
53	0.0944	0.0572	0.0485	0.0574	0.0717	0.0505	0.0597	0.0903
54	0.063	0.053	0.0677	0.0687	0.0605	0.0606	0.0565	0.0594
55	0.0786	0.05	0.0872	0.1107	0.0803	0.0622	0.6553	0.6416
56	0.0209	0.0182	0.0236	0.0253	0.0191	0.0189	0.02	0.0199
57	0.0114	0.0134	0.0086	0.008	0.0088	0.0111	0.0093	0.0092
58	0.0144	0.0138	0.0153	0.0169	0.0153	0.0135	0.0165	0.0186
59	0.0169	0.015	0.015	0.0164	0.0164	0.0141	0.0177	0.0185
60	0.0316	0.0222	0.0377	0.0439	0.0327	0.0281	0.0428	0.0355
61	0.0239	0.0221	0.0232	0.027	0.0234	0.0186	0.0237	0.021
62	0.0322	0.0144	0.0178	0.0196	0.017	0.0128	0.0188	0.019
63	0.0064	0.0041	0.0056	0.0066	0.0049	0.0038	0.0056	0.0057
64	0.4296	0.2064	0.3143	0.355	0.3061	0.2122	0.31	0.2966
65	0.1029	0.1081	0.1162	0.1278	0.1072	0.1013	0.1039	0.0946
66	0.0806	0.0713	0.05	0.056	0.0454	0.0488	0.044	0.0475
67	0.0715	0.0615	0.035	0.0387	0.0355	0.0386	0.0363	0.0386
68	0.0685	0.0314	0.0437	0.0512	0.0382	0.0317	0.0384	0.0343
69	0.0453	0.0567	0.0288	0.0263	0.0314	0.0412	0.032	0.039
70	0.1376	0.0417	0.0828	0.0937	0.0705	0.0502	0.065	0.0504
71	0.1489	0.0839	0.1104	0.1338	0.0995	0.0861	0.0959	0.0857
72	0.0811	0.0533	0.0777	0.0896	0.0643	0.0497		
73	0.0664	0.0403	0.0707	0.0838	0.0584	0.0444	0.0562	0.0409
74	0.0712	0.0496	0.0795	0.0901	0.0698	0.0555		
75	0.114	0.0756	0.1244	0.139	0.1049	0.0872	0.1081	0.0891
76	0.0354	0.0256	15.621	0.0297	0.0284	0.0238		
77	0.0186	0.0173	22.631	0.0178	0.019	0.0157	0.161	0.3622
78	0.0189	0.0157	11.687	0.0159	0.0164	0.014	0.0254	0.0656
79	0.0312	0.025	2.5151	0.027	0.0267	0.0245		
80	0.0538	0.0302	5.5879	0.039	0.0347	0.0272		

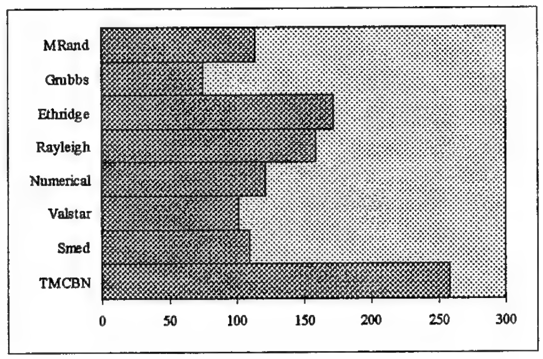
Design Point Reference Number	*Smed*	*Ethridge*	*MRand*	*Valstar*	*Grubbs*	*Rayleigh*	*Numerical*	*TMCBN*
81	0.0053	0.0048	0.6876	0.0057	0.0051	0.0052	0.0081	0.0096
82	0.01	0.0089	0.752	0.0086	0.0086	0.0088	0.0134	0.023
83	0.0154	0.0161	10.105	0.018	0.0176	0.0138	0.0281	0.0385
84	0.0176	0.018	0.1813	0.0199	0.0191	0.0155	0.0298	0.0384
85	0.0446	0.041	0.9366	0.0518	0.0464	0.0355	0.0708	0.0771
86	0.0405	0.0212	15.649	0.0272	0.0238	0.0218	262.52	375.79
87	0.0052	0.0047	0.3963	0.0052	0.005	0.0051	0.0079	0.0097
88	0.0095	0.0073	0.0763	0.0091	0.0082	0.0055	0.0132	0.0181
89	0.0203	0.0239	0.8131	0.025	0.0237	0.0218	0.0239	0.0285
90	0.1677	0.0971	0.2915	0.1346	0.1153	0.0967	0.1145	0.1036
91	0.09	0.0438	5.2747	0.0681	0.0554	0.0442	0.0584	0.0629
92	0.0664	0.0382	2.2842	0.0515	0.0464	0.0322	0.0463	0.0461
93	0.0166	0.0083	0.5928	0.0084	0.0082	0.0093	0.0083	0.0094
94	0.0451	0.0575	4.4111	0.0646	0.06	0.0517	0.0612	0.0661
95	0.1095	0.1068	4.3785	0.1326	0.1183	0.0969	0.1275	0.1383
96	0.0622	0.0401	29.806	0.0512	0.0438	0.0379	104.088	105.94
97	0.0493	0.0282	82.727	0.0419	0.0334	0.0245	0.0375	0.0451
98	0.0523	0.0353	7.8147	0.0468	0.04	0.0303	0.0418	0.0384
99	0.0604	0.0567	10.058	0.0705	0.062	0.0515		
100	0.0857	0.0728	3.815	0.0999	0.0842	0.0671		

Appendix I: Results of the Simulation Experiment Based on the Design Points

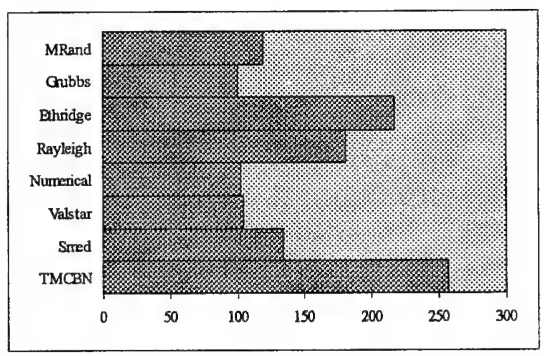
In each section, the tables indicate for each CEP estimator the number of design points that had the best (lowest) value for the MOEs considered in this study. The estimator with the best performance for a given factor level in the tables is shaded for quick interpretation of the results.

OVERALL RESULTS

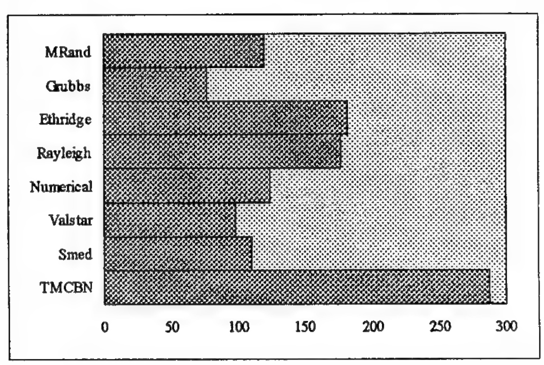
	MRE
TMCBN	258
Smed	109
Valstar	102
Numerical	121
Rayleigh	159
Ethridge	172
Grubbs	75
MRand	114



	VRE
TMCBN	257
Smed	134
Valstar	105
Numerical	103
Rayleigh	181
Ethridge	217
Grubbs	100
MRand	119



	MSRE
TMCBN	288
Smed	110
Valstar	98
Numerical	124
Rayleigh	177
Ethridge	182
Grubbs	77
MRand	120



DESIGN POINT MRE RESULTS

SAMPLE SIZE = 15

Overall

Smed	22
Ethridge	36
MRand	29
Valstar	27
Grubbs	21
Rayleigh	30
Numerical	47
TMCBN	69

Bias	0	0.5σ	1σ	2σ	4σ
Smed	0	1	2	9	10
Ethridge	0	10	10	10	6
MRand	5	13	8	3	0
Valstar	4	5	0	5	13
Grubbs	2	3	5	2	9
Rayleigh	2	5	10	12	1
Numerical	2	8	10	12	15
TMCBN	1	20	21	13	14

Correlation	-0.8	-0.4	0	0.4	0.8
Smed	2	2	5	5	8
Ethridge	8	5	5	6	12
MRand	4	6	9	4	6
Valstar	5	5	8	6	3
Grubbs	3	4	6	5	3
Rayleigh	0	4	12	8	6
Numerical	11	7	9	12	8
TMCBN	10	8	13	19	19

σ_y/σ_x	0.2	0.6	1	1.667	5
Smed	1	0	6	7	8
Ethridge	6	7	4	6	13
MRand	11	4	6	3	5
Valstar	0	14	8	5	0
Grubbs	6	6	1	4	4
Rayleigh	5	5	13	7	0
Numerical	9	9	9	10	10
TMCBN	17	12	9	15	16

SAMPLE SIZE = 9

Overall

Smed	38
Ethridge	33
MRand	24
Valstar	26
Grubbs	20
Rayleigh	40
Numerical	28
TMCBN	69

Bias	0	0.5σ	1σ	2σ	4σ
Smed	2	4	9	7	16
Ethridge	1	9	8	10	5
MRand	4	7	5	8	0
Valstar	1	5	4	2	14
Grubbs	1	9	4	1	5
Rayleigh	1	3	13	15	8
Numerical	0	3	4	11	10
TMCBN	5	25	18	11	10

Correlation	-0.8	-0.4	0	0.4	0.8
Smed	7	1	6	15	9
Ethridge	8	5	5	5	10
MRand	1	7	5	8	3
Valstar	3	5	7	4	7
Grubbs	5	1	6	4	4
Rayleigh	2	4	11	9	14
Numerical	5	9	7	4	3
TMCBN	9	8	19	17	16

σ_y/σ_x	0.2	0.6	1	1.667	5
Smed	4	6	8	8	12
Ethridge	11	5	4	4	9
MRand	8	5	3	4	4
Valstar	2	6	10	7	1
Grubbs	4	3	4	5	4
Rayleigh	6	7	11	10	6
Numerical	6	7	3	5	7
TMCBN	14	18	13	12	12

DESIGN POINT MRE RESULTS
(continued)

SAMPLE SIZE = 6

Overall

Smed	27
Ethridge	41
MRand	26
Valstar	24
Grubbs	15
Rayleigh	40
Numerical	29
TMCBN	74

Bias	0	0.5σ	1σ	2σ	4σ
Smed	0	7	2	9	9
Ethridge	2	10	13	10	6
MRand	3	8	5	10	0
Valstar	0	2	3	6	13
Grubbs	0	2	3	3	7
Rayleigh	2	4	8	12	14
Numerical	0	4	10	6	9
TMCBN	8	28	22	9	7

Correlation	-0.8	-0.4	0	0.4	0.8
Smed	3	3	8	5	8
Ethridge	8	5	9	8	11
MRand	2	4	7	5	8
Valstar	5	6	4	6	3
Grubbs	0	5	3	3	4
Rayleigh	5	2	11	12	10
Numerical	4	5	8	6	6
TMCBN	13	11	15	20	15

σ_y/σ_x	0.2	0.6	1	1.667	5
Smed	5	5	6	6	5
Ethridge	13	5	10	3	10
MRand	6	7	4	3	6
Valstar	1	9	3	10	1
Grubbs	2	4	1	6	2
Rayleigh	6	7	13	7	7
Numerical	6	3	3	4	13
TMCBN	16	16	15	16	11

SAMPLE SIZE = 3

Overall

Smed	22
Ethridge	62
MRand	35
Valstar	25
Grubbs	19
Rayleigh	49
Numerical	17
TMCBN	46

Bias	0	0.5σ	1σ	2σ	4σ
Smed	0	2	5	5	10
Ethridge	6	20	11	16	9
MRand	0	19	11	5	0
Valstar	0	2	2	4	17
Grubbs	1	1	3	4	10
Rayleigh	4	5	8	17	15
Numerical	0	0	6	10	1
TMCBN	4	16	19	4	3

Correlation	-0.8	-0.4	0	0.4	0.8
Smed	4	2	4	1	11
Ethridge	12	10	10	16	14
MRand	3	8	9	10	5
Valstar	4	5	9	4	3
Grubbs	1	1	4	9	4
Rayleigh	4	4	18	12	11
Numerical	7	3	1	2	4
TMCBN	5	7	10	11	13

σ_y/σ_x	0.2	0.6	1	1.667	5
Smed	3	2	3	8	6
Ethridge	16	9	5	9	23
MRand	4	8	12	7	4
Valstar	5	6	6	6	2
Grubbs	4	6	5	2	2
Rayleigh	6	9	12	11	11
Numerical	3	5	5	3	1
TMCBN	14	10	7	9	6

DESIGN POINT VRE RESULTS

SAMPLE SIZE = 15

Overall

Smed	29
Ethridge	55
MRand	31
Valstar	39
Grubbs	34
Rayleigh	37
Numerical	41
TMCBN	75

Bias	0	0.5σ	1σ	2σ	4σ
Smed	0	3	3	10	13
Ethridge	4	11	16	8	16
MRand	5	10	8	8	0
Valstar	2	6	5	6	20
Grubbs	1	3	8	5	17
Rayleigh	2	9	6	15	5
Numerical	0	6	8	10	17
TMCBN	1	21	18	13	22

Correlation	-0.8	-0.4	0	0.4	0.8
Smed	3	4	10	5	7
Ethridge	11	8	12	12	12
MRand	5	4	8	8	6
Valstar	5	10	10	11	3
Grubbs	8	7	4	8	7
Rayleigh	0	4	10	11	12
Numerical	11	6	7	11	6
TMCBN	18	8	13	17	19

σ_y/σ_x	0.2	0.6	1	1.667	5
Smed	4	2	9	4	10
Ethridge	13	6	12	10	14
MRand	10	8	5	5	3
Valstar	5	9	13	9	3
Grubbs	4	11	7	8	4
Rayleigh	4	10	3	17	3
Numerical	7	9	6	7	12
TMCBN	13	14	12	15	21

SAMPLE SIZE = 9

Overall

Smed	33
Ethridge	52
MRand	30
Valstar	27
Grubbs	26
Rayleigh	37
Numerical	27
TMCBN	68

Bias	0	0.5σ	1σ	2σ	4σ
Smed	1	0	9	11	12
Ethridge	4	19	15	4	10
MRand	2	12	7	9	0
Valstar	2	3	2	3	17
Grubbs	0	3	2	10	11
Rayleigh	1	11	5	11	9
Numerical	1	3	6	7	10
TMCBN	4	18	19	15	12

Correlation	-0.8	-0.4	0	0.4	0.8
Smed	8	5	4	7	9
Ethridge	7	11	8	15	11
MRand	5	6	6	6	7
Valstar	5	3	7	6	6
Grubbs	4	6	5	6	5
Rayleigh	2	3	15	9	8
Numerical	7	3	4	5	8
TMCBN	7	8	21	15	17

σ_y/σ_x	0.2	0.6	1	1.667	5
Smed	7	4	7	7	8
Ethridge	14	7	7	7	17
MRand	8	7	3	5	7
Valstar	1	7	6	7	6
Grubbs	6	6	6	7	1
Rayleigh	6	11	12	7	1
Numerical	5	7	4	3	8
TMCBN	13	14	14	15	12

DESIGN POINT VRE RESULTS
(continued)

SAMPLE SIZE = 6

<u>Overall</u>

Smed	41
Ethridge	55
MRand	26
Valstar	19
Grubbs	21
Rayleigh	47
Numerical	18
TMCBN	64

Bias	0	0.5σ	1σ	2σ	4σ
Smed	1	6	7	9	18
Ethridge	2	14	17	15	7
MRand	2	7	9	8	0
Valstar	2	3	1	2	11
Grubbs	1	2	4	4	10
Rayleigh	2	5	11	13	16
Numerical	0	2	1	8	7
TMCBN	5	26	16	10	7

Correlation	-0.8	-0.4	0	0.4	0.8
Smed	5	6	15	5	10
Ethridge	8	6	9	11	21
MRand	4	5	9	4	4
Valstar	4	2	5	4	4
Grubbs	4	6	5	4	2
Rayleigh	3	5	13	16	10
Numerical	4	2	4	3	5
TMCBN	11	9	11	19	14

σ_Y/σ_X	0.2	0.6	1	1.667	5
Smed	12	8	9	4	8
Ethridge	10	8	8	13	16
MRand	7	5	4	5	5
Valstar	1	6	2	6	4
Grubbs	2	6	5	7	1
Rayleigh	7	12	13	10	5
Numerical	2	4	5	3	4
TMCBN	16	13	12	8	15

SAMPLE SIZE = 3

<u>Overall</u>

Smed	31
Ethridge	55
MRand	32
Valstar	20
Grubbs	19
Rayleigh	60
Numerical	17
TMCBN	50

Bias	0	0.5σ	1σ	2σ	4σ
Smed	2	5	4	11	9
Ethridge	4	15	15	9	12
MRand	4	11	10	7	0
Valstar	0	3	4	3	10
Grubbs	0	5	1	4	9
Rayleigh	0	9	7	16	28
Numerical	2	0	6	5	4
TMCBN	3	18	18	10	1

Correlation	-0.8	-0.4	0	0.4	0.8
Smed	6	3	5	6	11
Ethridge	12	7	14	9	13
MRand	2	6	10	10	4
Valstar	3	5	5	1	6
Grubbs	4	3	3	6	3
Rayleigh	6	9	14	16	15
Numerical	3	2	6	4	2
TMCBN	8	5	11	14	12

σ_Y/σ_X	0.2	0.6	1	1.667	5
Smed	5	5	5	9	7
Ethridge	15	10	10	10	10
MRand	3	8	10	7	4
Valstar	2	5	4	5	4
Grubbs	1	6	4	5	3
Rayleigh	10	11	12	10	17
Numerical	3	2	6	4	2
TMCBN	16	12	6	6	10

DESIGN POINT MSRE RESULTS

SAMPLE SIZE = 15

Overall

Smed	22
Ethridge	34
MRand	30
Valstar	29
Grubbs	31
Rayleigh	34
Numerical	45
TMCBN	92

Bias	0	0.5σ	1σ	2σ	4σ
Smed	0	1	2	9	10
Ethridge	0	7	10	10	7
MRand	9	15	5	1	0
Valstar	4	5	2	6	12
Grubbs	1	4	5	5	16
Rayleigh	2	4	12	16	0
Numerical	0	8	8	10	19
TMCBN	2	26	25	19	20

Correlation	-0.8	-0.4	0	0.4	0.8
Smed	2	4	5	4	7
Ethridge	9	5	4	6	10
MRand	3	6	8	7	6
Valstar	3	7	9	7	3
Grubbs	5	6	8	6	6
Rayleigh	1	3	11	10	9
Numerical	10	5	11	13	6
TMCBN	17	10	17	22	26

σ_y/σ_x	0.2	0.6	1	1.667	5
Smed	2	1	7	6	6
Ethridge	6	4	7	5	12
MRand	11	5	5	3	6
Valstar	1	12	9	5	2
Grubbs	6	10	5	8	2
Rayleigh	5	7	12	9	1
Numerical	14	6	4	11	10
TMCBN	18	16	15	20	23

SAMPLE SIZE = 9

Overall

Smed	33
Ethridge	43
MRand	27
Valstar	26
Grubbs	15
Rayleigh	39
Numerical	34
TMCBN	73

Bias	0	0.5σ	1σ	2σ	4σ
Smed	1	1	5	8	18
Ethridge	2	15	9	9	8
MRand	3	10	7	7	0
Valstar	2	2	3	3	16
Grubbs	0	3	0	3	9
Rayleigh	2	4	13	16	4
Numerical	0	5	8	12	9
TMCBN	5	27	21	9	11

Correlation	-0.8	-0.4	0	0.4	0.8
Smed	8	3	6	8	8
Ethridge	8	8	8	9	10
MRand	3	7	7	7	3
Valstar	4	4	8	5	5
Grubbs	2	2	0	8	3
Rayleigh	0	3	13	8	15
Numerical	10	9	7	3	5
TMCBN	8	9	18	19	19

σ_y/σ_x	0.2	0.6	1	1.667	5
Smed	5	6	5	8	9
Ethridge	13	4	7	6	13
MRand	9	6	4	5	3
Valstar	3	7	8	6	2
Grubbs	2	4	5	2	2
Rayleigh	5	9	12	9	4
Numerical	10	5	6	3	10
TMCBN	13	16	13	17	14

DESIGN POINT MSRE RESULTS
(continued)

SAMPLE SIZE = 6

Overall

Smed	31
Ethridge	45
MRand	26
Valstar	23
Grubbs	16
Rayleigh	40
Numerical	30
TMCBN	80

Bias	0	0.5σ	1σ	2σ	4σ
Smed	0	6	6	7	12
Ethridge	2	12	15	12	4
MRand	3	9	6	8	0
Valstar	1	1	1	4	16
Grubbs	0	1	1	5	9
Rayleigh	2	7	7	14	10
Numerical	0	2	8	9	11
TMCBN	7	30	22	9	12

Correlation	-0.8	-0.4	0	0.4	0.8
Smed	4	4	11	4	8
Ethridge	12	3	6	10	14
MRand	2	3	9	6	6
Valstar	4	3	5	7	4
Grubbs	0	5	3	6	2
Rayleigh	1	3	13	13	10
Numerical	9	5	5	5	6
TMCBN	11	14	16	19	20

σ_y/σ_x	0.2	0.6	1	1.667	5
Smed	6	7	7	5	6
Ethridge	11	5	6	10	13
MRand	8	7	3	3	5
Valstar	2	6	3	6	6
Grubbs	4	3	3	5	1
Rayleigh	7	9	11	9	4
Numerical	6	3	6	7	8
TMCBN	12	16	16	20	16

SAMPLE SIZE = 3

Overall

Smed	24
Ethridge	60
MRand	37
Valstar	20
Grubbs	15
Rayleigh	64
Numerical	15
TMCBN	43

Bias	0	0.5σ	1σ	2σ	4σ
Smed	0	2	5	7	10
Ethridge	6	19	14	12	9
MRand	2	19	11	5	0
Valstar	0	1	1	6	12
Grubbs	0	1	3	1	10
Rayleigh	2	7	9	23	23
Numerical	0	1	5	8	1
TMCBN	5	15	17	4	2

Correlation	-0.8	-0.4	0	0.4	0.8
Smed	4	1	4	5	10
Ethridge	13	9	9	15	14
MRand	5	9	8	9	6
Valstar	3	2	10	2	3
Grubbs	3	2	2	6	2
Rayleigh	4	9	22	15	14
Numerical	6	3	1	1	4
TMCBN	3	5	10	12	13

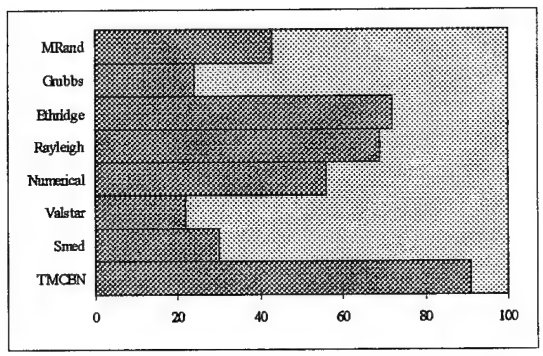
σ_y/σ_x	0.2	0.6	1	1.667	5
Smed	3	3	3	9	6
Ethridge	16	9	7	11	17
MRand	5	9	9	7	7
Valstar	1	5	5	7	2
Grubbs	3	5	4	0	3
Rayleigh	9	13	16	12	14
Numerical	2	4	4	3	2
TMCBN	16	8	7	7	5

Appendix J: Results of the Simulation Experiment Based on the Sample Analysis Sets

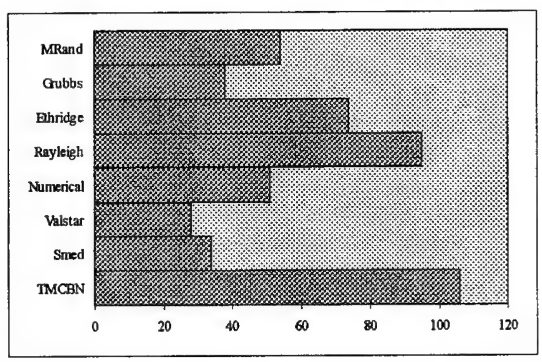
Like Appendix I, in this appendix the tables indicate for each CEP estimator the number of design points that had the best (lowest) value for the MOEs considered in this study. The estimator with the best performance for a given factor level in the tables is shaded for quick interpretation of the results.

OVERALL RESULTS

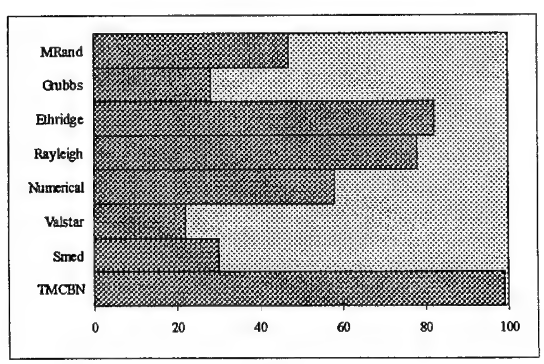
	MRE
TMCBN	91
Smed	30
Valstar	22
Numerical	56
Rayleigh	69
Ethridge	72
Grubbs	24
MRand	43



	VRE
TMCBN	106
Smed	34
Valstar	28
Numerical	51
Rayleigh	95
Ethridge	74
Grubbs	38
MRand	54



	MSRE
TMCBN	99
Smed	30
Valstar	22
Numerical	58
Rayleigh	78
Ethridge	82
Grubbs	28
MRand	47



SAMPLE ANALYSIS SET
MRE RESULTS

SAMPLE SIZE = 15

Overall

Smed	6
Ethridge	9
MRand	10
Valstar	6
Grubbs	10
Rayleigh	11
Numerical	25
TMCBN	24

bîas	[0, 0.75 $\bar{\sigma}$]	(0.75 $\bar{\sigma}$, 1.25 $\bar{\sigma}$]	(1.25 $\bar{\sigma}$, 2.75 $\bar{\sigma}$]	(>2.75 $\bar{\sigma}$)
Smed	0	0	3	3
Ethridge	2	5	2	0
MRand	9	1	0	0
Valstar	1	0	0	5
Grubbs	0	2	4	4
Rayleigh	4	4	3	0
Numerical	5	3	6	11
TMCBN	4	11	7	2

$\bar{\rho}$	[-1,-0.6)	[-0.6,-0.2)	[-0.2,0.2]	(0.2,0.6]	(0.6,1]
Smed	2	0	1	2	1
Ethridge	2	1	0	4	2
MRand	3	3	1	1	2
Valstar	1	1	0	4	0
Grubbs	2	2	4	2	0
Rayleigh	1	2	4	2	2
Numerical	4	7	8	1	5
TMCBN	5	5	2	4	8

Sy/Sx	(<0.4)	[0.4,0.8)	[0.8,1.25]	(1.25,2.5]	(>2.5)
Smed	1	0	1	1	3
Ethridge	0	0	1	2	6
MRand	5	2	2	1	0
Valstar	0	4	1	1	0
Grubbs	1	4	2	3	0
Rayleigh	1	1	5	4	0
Numerical	6	4	3	7	5
TMCBN	6	6	5	1	6

SAMPLE SIZE = 9

Overall

Smed	5
Ethridge	19
MRand	7
Valstar	3
Grubbs	8
Rayleigh	12
Numerical	17
TMCBN	29

bîas	[0, 0.75 $\bar{\sigma}$]	(0.75 $\bar{\sigma}$, 1.25 $\bar{\sigma}$]	(1.25 $\bar{\sigma}$, 2.75 $\bar{\sigma}$]	(>2.75 $\bar{\sigma}$)
Smed	0	1	0	4
Ethridge	3	3	8	5
MRand	5	0	2	0
Valstar	0	1	0	2
Grubbs	4	1	1	2
Rayleigh	3	4	2	3
Numerical	3	6	2	6
TMCBN	7	9	10	3

$\bar{\rho}$	[-1,-0.6)	[-0.6,-0.2)	[-0.2,0.2]	(0.2,0.6]	(0.6,1]
Smed	0	0	2	1	2
Ethridge	5	5	1	4	4
MRand	1	3	1	1	1
Valstar	1	0	2	0	0
Grubbs	1	2	5	0	0
Rayleigh	1	3	2	5	1
Numerical	8	4	0	1	4
TMCBN	3	3	7	8	8

Sy/Sx	(<0.4)	[0.4,0.8)	[0.8,1.25]	(1.25,2.5]	(>2.5)
Smed	0	2	1	0	2
Ethridge	4	2	1	5	7
MRand	2	1	2	0	2
Valstar	0	0	2	1	0
Grubbs	2	1	2	1	2
Rayleigh	2	2	4	3	1
Numerical	3	5	3	3	3
TMCBN	7	7	5	7	3

SAMPLE ANALYSIS SET
MRE RESULTS (continued)

SAMPLE SIZE = 6

Overall

Smed	7
Ethridge	20
MRand	10
Valstar	5
Grubbs	3
Rayleigh	18
Numerical	12
TMCBN	29

bîas	[0, 0.75 $\bar{\sigma}$]	(0.75 $\bar{\sigma}$, 1.25 $\bar{\sigma}$]	(1.25 $\bar{\sigma}$, 2.75 $\bar{\sigma}$]	(>2.75 $\bar{\sigma}$)
Smed	0	2	1	4
Ethridge	1	1	8	10
MRand	8	2	0	0
Valstar	3	1	0	1
Grubbs	0	0	1	2
Rayleigh	4	4	4	6
Numerical	2	5	1	4
TMCBN	9	10	10	0

$\bar{\rho}$	[-1,-0.6)	[-0.6,-0.2)	[-0.2,0.2]	(0.2,0.6]	(0.6,1]
Smed	3	3	1	0	0
Ethridge	2	7	2	4	5
MRand	4	2	1	3	0
Valstar	3	1	0	1	0
Grubbs	0	1	1	1	0
Rayleigh	1	2	8	4	3
Numerical	1	1	4	4	2
TMCBN	6	5	3	5	10

Sy/Sx	(<0.4)	[0.4,0.8)	[0.8,1.25]	(1.25,2.5]	(>2.5)
Smed	2	1	4	0	0
Ethridge	3	4	4	3	6
MRand	2	5	1	0	2
Valstar	0	0	2	3	0
Grubbs	0	2	0	1	0
Rayleigh	1	6	7	1	3
Numerical	4	1	1	3	3
TMCBN	8	4	2	9	6

SAMPLE SIZE = 3

Overall

Smed	12
Ethridge	24
MRand	16
Valstar	8
Grubbs	3
Rayleigh	28
Numerical	2
TMCBN	9

bîas	[0, 0.75 $\bar{\sigma}$]	(0.75 $\bar{\sigma}$, 1.25 $\bar{\sigma}$]	(1.25 $\bar{\sigma}$, 2.75 $\bar{\sigma}$]	(>2.75 $\bar{\sigma}$)
Smed	3	6	0	3
Ethridge	3	3	12	6
MRand	9	6	1	0
Valstar	3	2	2	1
Grubbs	0	0	1	2
Rayleigh	3	6	6	13
Numerical	1	0	1	0
TMCBN	3	3	2	1

$\bar{\rho}$	[-1,-0.6)	[-0.6,-0.2)	[-0.2,0.2]	(0.2,0.6]	(0.6,1]
Smed	2	2	5	3	0
Ethridge	7	3	3	5	6
MRand	6	5	1	3	1
Valstar	2	3	2	1	0
Grubbs	0	3	0	0	0
Rayleigh	4	5	7	4	8
Numerical	0	0	0	1	1
TMCBN	0	0	2	3	4

Sy/Sx	(<0.4)	[0.4,0.8)	[0.8,1.25]	(1.25,2.5]	(>2.5)
Smed	0	1	5	5	1
Ethridge	6	2	6	3	7
MRand	4	7	1	1	3
Valstar	1	2	1	4	0
Grubbs	2	1	0	0	0
Rayleigh	4	6	7	5	6
Numerical	1	0	0	1	0
TMCBN	3	1	1	1	3

SAMPLE ANALYSIS SET
VRE RESULTS

SAMPLE SIZE = 15

Overall

Smed	6
Ethridge	17
MRand	14
Valstar	9
Grubbs	16
Rayleigh	14
Numerical	28
TMCBN	37

bîas	[0, 0.75 $\bar{\sigma}$]	(0.75 $\bar{\sigma}$, 1.25 $\bar{\sigma}$]	(1.25 $\bar{\sigma}$, 2.75 $\bar{\sigma}$]	(>2.75 $\bar{\sigma}$)
Smed	0	0	1	5
Ethridge	2	7	3	5
MRand	8	4	2	0
Valstar	1	1	1	6
Grubbs	4	1	5	6
Rayleigh	4	2	7	1
Numerical	5	3	8	12
TMCBN	9	10	10	8

$\bar{\rho}$	[-1,-0.6)	[-0.6,-0.2)	[-0.2,0.2]	(0.2,0.6]	(0.6,1]
Smed	2	0	0	2	2
Ethridge	5	5	0	2	5
MRand	2	4	3	3	2
Valstar	2	3	1	3	0
Grubbs	3	5	3	4	1
Rayleigh	2	3	6	2	1
Numerical	9	7	5	5	2
TMCBN	7	5	7	9	9

Sy/Sx	(<0.4)	[0.4,0.8)	[0.8,1.25]	(1.25,2.5]	(>2.5)
Smed	1	0	1	1	3
Ethridge	3	0	3	3	8
MRand	4	2	3	3	2
Valstar	2	2	3	2	0
Grubbs	2	4	8	1	1
Rayleigh	2	4	4	4	0
Numerical	4	6	9	5	4
TMCBN	8	9	3	11	6

SAMPLE SIZE = 9

Overall

Smed	4
Ethridge	19
MRand	10
Valstar	5
Grubbs	9
Rayleigh	25
Numerical	12
TMCBN	30

bîas	[0, 0.75 $\bar{\sigma}$]	(0.75 $\bar{\sigma}$, 1.25 $\bar{\sigma}$]	(1.25 $\bar{\sigma}$, 2.75 $\bar{\sigma}$]	(>2.75 $\bar{\sigma}$)
Smed	0	1	1	2
Ethridge	2	4	7	6
MRand	9	0	1	0
Valstar	2	1	2	0
Grubbs	5	0	0	4
Rayleigh	2	5	7	11
Numerical	5	3	1	3
TMCBN	9	12	7	2

$\bar{\rho}$	[-1,-0.6)	[-0.6,-0.2)	[-0.2,0.2]	(0.2,0.6]	(0.6,1]
Smed	0	1	1	1	1
Ethridge	8	2	1	4	4
MRand	2	2	2	2	2
Valstar	1	3	1	0	0
Grubbs	2	3	0	3	1
Rayleigh	2	6	8	8	1
Numerical	5	3	1	2	1
TMCBN	2	5	6	7	10

Sy/Sx	(<0.4)	[0.4,0.8)	[0.8,1.25]	(1.25,2.5]	(>2.5)
Smed	0	1	1	1	1
Ethridge	4	1	2	4	8
MRand	1	1	3	3	2
Valstar	0	1	3	1	0
Grubbs	2	1	4	1	1
Rayleigh	5	7	5	5	3
Numerical	2	1	4	2	3
TMCBN	7	7	6	5	5

SAMPLE ANALYSIS SET
VRE RESULTS (continued)

SAMPLE SIZE = 6

Overall

Smed	11
Ethridge	22
MRand	10
Valstar	3
Grubbs	7
Rayleigh	24
Numerical	6
TMCBN	27

bîas	[0, 0.75 $\bar{\sigma}$]	(0.75 $\bar{\sigma}$, 1.25 $\bar{\sigma}$]	(1.25 $\bar{\sigma}$, 2.75 $\bar{\sigma}$]	(>2.75 $\bar{\sigma}$)
Smed	2	2	1	6
Ethridge	0	2	12	8
MRand	8	2	0	0
Valstar	3	0	0	0
Grubbs	1	3	1	2
Rayleigh	3	3	6	12
Numerical	1	4	1	0
TMCBN	10	12	5	0

$\bar{\rho}$	[-1,-0.6)	[-0.6,-0.2)	[-0.2,0.2]	(0.2,0.6]	(0.6,1]
Smed	3	3	3	2	0
Ethridge	3	6	3	4	6
MRand	3	2	3	1	1
Valstar	1	1	0	1	0
Grubbs	4	0	2	1	0
Rayleigh	4	4	8	5	3
Numerical	1	0	3	2	0
TMCBN	4	5	1	6	11

Sy/Sx	(<0.4)	[0.4,0.8)	[0.8,1.25]	(1.25,2.5]	(>2.5)
Smed	3	1	4	2	1
Ethridge	4	4	3	3	8
MRand	1	4	2	0	3
Valstar	0	0	1	2	0
Grubbs	0	3	3	1	0
Rayleigh	3	7	6	4	4
Numerical	1	1	1	2	1
TMCBN	8	4	3	7	5

SAMPLE SIZE = 3

Overall

Smed	13
Ethridge	16
MRand	20
Valstar	11
Grubbs	6
Rayleigh	32
Numerical	5
TMCBN	12

bîas	[0, 0.75 $\bar{\sigma}$]	(0.75 $\bar{\sigma}$, 1.25 $\bar{\sigma}$]	(1.25 $\bar{\sigma}$, 2.75 $\bar{\sigma}$]	(>2.75 $\bar{\sigma}$)
Smed	4	6	1	2
Ethridge	1	2	10	3
MRand	12	7	1	0
Valstar	3	5	2	1
Grubbs	2	2	0	2
Rayleigh	1	4	9	18
Numerical	2	2	0	1
TMCBN	4	6	2	0

$\bar{\rho}$	[-1,-0.6)	[-0.6,-0.2)	[-0.2,0.2]	(0.2,0.6]	(0.6,1]
Smed	1	1	7	3	1
Ethridge	3	3	2	2	6
MRand	7	6	4	2	1
Valstar	1	5	3	2	0
Grubbs	0	4	1	1	0
Rayleigh	6	6	9	6	5
Numerical	0	0	1	3	1
TMCBN	2	0	1	3	6

Sy/Sx	(<0.4)	[0.4,0.8)	[0.8,1.25]	(1.25,2.5]	(>2.5)
Smed	0	0	5	5	3
Ethridge	4	3	2	3	4
MRand	6	4	3	3	4
Valstar	0	5	0	5	1
Grubbs	0	2	1	3	0
Rayleigh	5	7	8	7	5
Numerical	1	0	2	2	0
TMCBN	4	2	1	2	3

SAMPLE ANALYSIS SET
MSRE RESULTS

SAMPLE SIZE = 15

Overall

Smed	6
Ethridge	16
MRand	12
Valstar	6
Grubbs	13
Rayleigh	10
Numerical	29
TMCBN	35

bîas	[0, 0.75 $\bar{\sigma}$]	(0.75 $\bar{\sigma}$, 1.25 $\bar{\sigma}$]	(1.25 $\bar{\sigma}$, 2.75 $\bar{\sigma}$]	(>2.75 $\bar{\sigma}$)
Smed	0	0	3	3
Ethridge	3	5	3	5
MRand	9	2	1	0
Valstar	1	1	0	4
Grubbs	2	0	3	8
Rayleigh	2	3	5	0
Numerical	6	3	7	13
TMCBN	9	11	9	6

$\bar{\rho}$	[-1,-0.6)	[-0.6,-0.2)	[-0.2,0.2]	(0.2,0.6]	(0.6,1]
Smed	1	0	1	3	1
Ethridge	5	4	1	2	4
MRand	2	4	2	2	2
Valstar	1	1	1	3	0
Grubbs	0	4	5	3	1
Rayleigh	1	2	4	0	3
Numerical	9	8	7	2	3
TMCBN	6	6	6	8	9

Sy/Sx	(<0.4)	[0.4,0.8)	[0.8,1.25]	(1.25,2.5]	(>2.5)
Smed	1	0	1	0	4
Ethridge	2	0	2	4	8
MRand	4	2	3	2	1
Valstar	0	3	2	1	0
Grubbs	1	5	3	4	0
Rayleigh	2	2	3	3	0
Numerical	5	8	5	8	3
TMCBN	9	6	5	9	6

SAMPLE SIZE = 9

Overall

Smed	4
Ethridge	21
MRand	9
Valstar	3
Grubbs	7
Rayleigh	19
Numerical	19
TMCBN	26

bîas	[0, 0.75 $\bar{\sigma}$]	(0.75 $\bar{\sigma}$, 1.25 $\bar{\sigma}$]	(1.25 $\bar{\sigma}$, 2.75 $\bar{\sigma}$]	(>2.75 $\bar{\sigma}$)
Smed	0	1	0	3
Ethridge	3	2	8	8
MRand	8	0	1	0
Valstar	1	0	2	0
Grubbs	1	1	0	5
Rayleigh	2	5	5	7
Numerical	7	4	2	6
TMCBN	6	12	7	1

$\bar{\rho}$	[-1,-0.6)	[-0.6,-0.2)	[-0.2,0.2]	(0.2,0.6]	(0.6,1]
Smed	0	0	1	1	2
Ethridge	8	4	1	4	4
MRand	2	1	3	2	1
Valstar	1	2	0	0	0
Grubbs	0	4	1	1	1
Rayleigh	2	3	6	6	2
Numerical	5	6	3	1	4
TMCBN	3	6	5	6	6

Sy/Sx	(<0.4)	[0.4,0.8)	[0.8,1.25]	(1.25,2.5]	(>2.5)
Smed	0	2	1	0	1
Ethridge	4	1	1	6	9
MRand	2	1	2	3	1
Valstar	0	1	2	0	0
Grubbs	1	1	3	1	1
Rayleigh	3	6	4	4	2
Numerical	4	4	6	1	4
TMCBN	6	5	6	5	4

SAMPLE ANALYSIS SET
MSRE RESULTS (continued)

SAMPLE SIZE = 6

<u>Overall</u>

Smed	9
Ethridge	25
MRand	7
Valstar	4
Grubbs	3
Rayleigh	19
Numerical	8
TMCBN	28

bîas	$[0, 0.75\bar{\sigma}]$	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	$(>2.75\bar{\sigma})$
Smed	1	2	1	5
Ethridge	0	1	11	13
MRand	5	2	0	0
Valstar	3	0	0	1
Grubbs	0	1	1	1
Rayleigh	4	3	5	7
Numerical	1	5	2	0
TMCBN	11	11	6	0

$\bar{\rho}$	[-1,-0.6)	[-0.6,-0.2)	[-0.2,0.2]	(0.2,0.6]	(0.6,1]
Smed	3	3	2	1	0
Ethridge	4	6	3	6	6
MRand	3	1	1	2	0
Valstar	2	1	0	1	0
Grubbs	1	0	2	0	0
Rayleigh	1	4	7	4	3
Numerical	3	1	3	1	0
TMCBN	4	5	3	5	11

Sy/Sx	(<0.4)	[0.4,0.8)	[0.8,1.25]	(1.25,2.5]	(>2.5)
Smed	3	1	4	1	0
Ethridge	5	6	3	6	5
MRand	1	4	1	0	1
Valstar	0	0	2	2	0
Grubbs	0	2	1	0	0
Rayleigh	0	4	8	2	5
Numerical	3	0	1	2	2
TMCBN	8	4	2	7	7

SAMPLE SIZE = 3

<u>Overall</u>

Smed	11
Ethridge	20
MRand	19
Valstar	9
Grubbs	5
Rayleigh	30
Numerical	2
TMCBN	10

bîas	$[0, 0.75\bar{\sigma}]$	$(0.75\bar{\sigma}, 1.25\bar{\sigma}]$	$(1.25\bar{\sigma}, 2.75\bar{\sigma}]$	$(>2.75\bar{\sigma})$
Smed	2	7	0	2
Ethridge	2	2	12	4
MRand	12	5	2	0
Valstar	3	3	2	1
Grubbs	2	1	0	2
Rayleigh	2	4	7	17
Numerical	1	0	1	0
TMCBN	4	5	1	0

$\bar{\rho}$	[-1,-0.6)	[-0.6,-0.2)	[-0.2,0.2]	(0.2,0.6]	(0.6,1]
Smed	1	2	5	3	0
Ethridge	4	3	3	3	7
MRand	7	6	3	2	1
Valstar	1	5	2	1	0
Grubbs	0	3	0	2	0
Rayleigh	6	6	7	4	7
Numerical	0	0	0	1	1
TMCBN	1	0	1	4	4

Sy/Sx	(<0.4)	[0.4,0.8)	[0.8,1.25]	(1.25,2.5]	(>2.5)
Smed	0	0	5	4	2
Ethridge	7	2	3	3	5
MRand	4	6	3	2	4
Valstar	0	5	0	4	0
Grubbs	1	2	1	1	0
Rayleigh	4	7	7	6	6
Numerical	1	0	0	1	0
TMCBN	3	1	1	1	4

Bibliography

Asner, Bernie (1993), "CEP Definition and Calculation", Sverdrop Technology, P.O. Box 1935, Eglin AFB, FL 32542-5000.

Astbury, K. A. (1982), "Computing CEP According to the Popular Approximation Formulas", TRW Interoffice Correspondence.

CACI Products Company (1995), *MODSIM II: The Language for Object Oriented Programming*, 3333 North Torrey Pines Court, La Jolle, CA 92307.

Clemen, Robert T. (1996), *Making Hard Decisions*, Wadsworth Publishing Company, 10 Davis Drive, Belmont, CA 94002.

Elder, Richard L. (1986), "An Examination of Circular Error Probable Approximation Techniques", AFIT/GST/ENS/86M-6.

Ethridge, Ronald A. (1983), "Robust Estimation of Circular Error Probable For Small Samples", Air Command and Staff College, Maxwell AFB, AL.

Graybill, Franklin A. (1976), *Theory and Application of the Linear Model*, Duxbury Press, Boston, MA.

Grubbs, Frank E. (1964), "Approximate Circular and Noncircular Offset Probabilities of Hitting", *Operations Research, 12*, pages 51-62.

Harter, H. Leon (1960), "Circular Error Probabilities", *American Statistical Association Journal, 55*: pages 723-731.

Kamat, A. R. (1962), "Some More Estimates of Circular Error Probable", *American Statistical Association Journal, 57*: pages 191-195.

Lewis, Charles L. (1994), "Estimation techniques for Guided Rocket Probability of Hit, Cost/Kill, and Stowed Kills Against Point and Small Area Targets", Redstone Arsenal, AL, 35898-5000.

Mathsoft, Inc. (1995), *MathCAD User's Guide, version 6.0*, One Kendall Square, Cambridge, MA 02139..

Nelson, William (1988), "Use of Circular Error Probability in Target Detection", Mitre Corporation, Bedford, MA.

Neter, J., Kutner, M., Nachtshiem, C., and Wasserman, W. (1996), *Applied Linear Statistical Models*, Irwin Co., Burr Ridge, IL.

Pesapane, John and Irvine, Robert B. (1977), "Derivation of the CEP Formula to Approximate Rand-234 CEP Tables", SAC Interoffice Correspondence.

Puhek, Peter (1992), "A Sensitivity Analysis of Circular Error Probable Approximation Techniques", AFIT/GOR/ENS/92M-23.

Rand R-234 (1952), *Offset Circle Probabilities,* Rand Corporation, 1700 Main Street, Santa Monica, CA.

Statistix (1985), *Statistix Users Guide, version 1.0,* P.O. Box 12815, Tallahassee, Fl 32317-2185

Smith, C. C. (1982), "Methods of CEP Calculation", TRW Interoffice Correspondence.

Thacker, James R. and Tucker, Shirley J. (1983), "CEP Tabular Estimation For Biased and Elliptical Distributions", Redstone Arsenal, AL 35898-5000.

Thompson, David D. (1964), "Development of CEP Formula", SAC Memo for Record.

Tongue, William L. (1993), "An Empirical Evaluation of Five Circular Error Probable Estimation Techniques and a Method for Improving Them", AFIT/GST/ENS/93M.

Vita

Capt. Charles E. Williams was born on 11 May 1961 in Pittsburg, Kansas. He graduated Valedictorian from Liberal High School in Liberal, Missouri in 1979 and graduated from Missouri Southern State College with a B.S. degree in Teaching Secondary Mathematics/Physics in 1984. After working for the Jasper County Highway Department for 1 year, Charles spent 4 years as a high school mathematics/physics instructor and girls basketball coach before entering the USAF in 1989. While serving as a missile combat crewmember at Whiteman AFB, Missouri from 1990-1993, he completed a M.S. degree in Mathematics at nearby Central Missouri State College. After serving as a missile combat crewmember in Grand Forks AFB, North Dakota in 1994-1995, he entered the Graduate of Operations Analysis Program, School of Engineering, Air Force Institute of Technology, Wright-Patterson AFB, Ohio.

Charles married Chong Hui Capan in 1988. The couple has one daughter, Kimberly Ann Williams.

Permanent Address: 2724 N. St. Charles Ave.
Joplin, MO 64801

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	March 1997	Masters Thesis

4. TITLE AND SUBTITLE
A Comparison of Circular Error Probable Estimators for Small Samples

5. FUNDING NUMBERS

6. AUTHOR(S)
Charles E. Williams

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Air Force Institute of Technology
WPAFB, OH 45433-6583

8. PERFORMING ORGANIZATION REPORT NUMBER
AFIT/GOA/ENS/97M-14

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Distribution Unlimited

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

Several previous studies investigated the performance of competing circular error probable (CEP) estimators for small samples. This estimation is important in ICBM analysis because, due to expense, there are a limited number of ICBM test launches. In the most recent of of these previous studies, Tongue (1993) considered five CEP estimators in a simulation test. In this paper, Tongue's findings are built on in three ways:

1. The number of estimators compared is expanded to eight.
2. Different factors and factor levels are used.
3. In addition to analyzing simulated samples based on the population parameters used to create them, we sort the entire set of samples generated into subsets (sample analysis sets) based strictly on sample statistics. Analysis on these sample analysis sets models the real life situation of estimating CEP, given a small sample from an unknown population, using only sample statistics.

14. SUBJECT TERMS
Circular Error Probable, CEP

15. NUMBER OF PAGES
198

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL